Physikalisches Institut
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Theoretische Physik

Exercise Sheet 3
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SS 16

Exercises General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/klemm/grss16/

–Homework–

1 Getting used to manifolds (10 points)

In the lecture the concept of manifolds has been introduced. This exercise is devoted to build up some intuition for these objects. To this end, let us first recall the definition: A differentiable manifold of dimension n is a topological space[1] M, such that the space can be covered with a set of open sets $\{U_\alpha\}$ and for every α we have a diffeomorphism[2] $h_\alpha : U_\alpha \longrightarrow V_\alpha$ to an open set $V_\alpha \subset \mathbb{R}^n$. A pair (U_α, h_α) is called a chart and a set of charts covering M is called an atlas. Further, for every pair (α, β) such that $U_\alpha \cap U_\beta \neq \emptyset$ the transition function $h_{\alpha\beta} \equiv h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \longrightarrow h_\alpha(U_\alpha \cap U_\beta)$ is required to be a diffeomorphism.

a) Consider $\mathbb{R}^n$ itself as a topological space and explicitly construct an atlas to show that it is a manifold. (*2 points*)

   Hint: One chart is enough.

This result is not suprising since manifolds are constructed such that they locally look like the $\mathbb{R}^n$ and of course does $\mathbb{R}^n$ locally look like $\mathbb{R}^n$.
A manifold is called topologically trivial if it can be continuously shrunk to a point, an example is $\mathbb{R}^n$. One may think that the property of $\mathbb{R}^n$ being topologically trivial allowed for covering it with one chart only. This is not true, as we will see in the next item.

b) Consider the infinitely long cyclinder M given by its embedding in $\mathbb{R}^3$,

$$M = \{(R\cos\phi, R\sin\phi, t) \,|\, \phi \in [0, 2\pi),\, t \in (-\infty, \infty),\, R > 0\} \,. \tag{1}$$

   Although M is topologically non-trivial — it can be shrunk to a circle but not to a point — it can be covered with a single chart only. Construct such a chart explitly. (*4 points*)

   Hint: It may be easier to first think about how M can be mapped to the punctured complex plane, $\mathbb{C} \setminus \{0\}$.

c) Now consider the two-dimensional torus given by its embedding in $\mathbb{R}^3$,

$$T^2 = \{((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta) \,|\, \theta, \phi \in [0, 2\pi),\ R > r > 0\} \,. \tag{2}$$

   Explicitly construct an atlas to show that T^2 is a manifold. (*4 points*)

[1] This topological space has to fulfill certain technical assumptions to allow for a sensible calculus on them, but these we do not consider here.

[2] A diffeomorphism is a homeomorphism with the additional property that it and its inverse are continuously differentiable.

2 Induced metrics and de Sitter space (10 points)

A manifold M can be given an additional structure by endowing it with a metric tensor, which provides a natural generalization of the scalar product between two vectors in $\mathbb{R}^n$. In terms of the language introduced on sheet 3, a metric tensor g is a smooth tensor field of type $(0,2)$ on M that is symmetric and non-degnerate. This means:

- For every $p \in M$ there is a tensor g_p of type $(0,2)$ on $V = T_pM$
- for every $u, v \in T_pM$ the equality $g_p(u,v) = g_p(v,u)$ holds
- if $g_p(u,v) = 0$ for every $v \in T_pM$ then $u = 0$
- the map $p \longmapsto g_p$ is smooth.

In local coordinates $\{x^\mu\}$ on M the metric tensor can be expanded as

$$g = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{3}$$

in terms of smooth functions $g_{\mu\nu}$. Examples are $\mathbb{R}^n$ equipped with the standard euclidean metric, which is just what we call $\mathbb{R}^n$, or $\mathbb{R}^n$ equipped with the Minkowski metric, which is n-dimensional Minkowski space $\mathbb{R}^{1,n-1}$. This also shows that one and the same manifold can be equipped with different metrics, by which it is made into different objects.
Another important concept is that of the *pullback*. Consider a manifold M equipped with a metric g and a second manifold N. If we further have a smooth map $\phi : N \longrightarrow M$ we can use this map to pull g back onto N. This gives the so called *induced metric* on N, which is denoted as $\phi^* g$. With g as in eq. (3) and with local coordinates $\{y^\mu\}$ on N the induced metric locally reads

$$\phi^* g = \left[g_{\alpha\beta} \left(\frac{\partial \phi^\alpha}{\partial y^\mu} \right) \left(\frac{\partial \phi^\beta}{\partial y^\nu} \right) \right] \mathrm{d}y^\mu \otimes \mathrm{d}y^\nu \ . \tag{4}$$

a) Consider the two-sphere S^2 embedded in $\mathbb{R}^3$,

$$S^2 = \{ R(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \,|\, \phi \in [0, 2\pi),\ \theta \in [0,\pi),\ R > 0 \}. \tag{5}$$

Use the inclusion map

$$\begin{aligned} \iota : \qquad S^2 &\longrightarrow \mathbb{R}^3 \\ (\theta, \phi) &\longmapsto R(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \end{aligned} \tag{6}$$

to calculate the induced metric on S^2. *(3 points)*

b) The embedding of T^2 in $\mathbb{R}^3$ is given in eq. (2). Use the corresponding inclusion map to calculate the induced metric on T^2. *(3 points)*

c) In cosmology the so called *de Sitter space* will be of importance. This space is cut out of five-dimensional Minkowski space $\mathbb{R}^{1,4}$ — with coordinates u, w, x, y, z, with u being timelike — by the hyperboloid equation

$$-u^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2, \quad \alpha \in \mathbb{R} \ . \tag{7}$$

On de Sitter space we introduce coordinates t, χ, θ, ϕ and embed it in $\mathbb{R}^{1,4}$ by

$$
\begin{aligned}
u &= \alpha \sinh(t/\alpha) \\
w &= \alpha \cosh(t/\alpha) \cos\chi \\
x &= \alpha \cosh(t/\alpha) \sin\chi \cos\theta \\
y &= \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\phi \\
z &= \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\phi \ .
\end{aligned}
\tag{8}
$$

Calculate the induced metric on de Sitter space. *(4 points)*