Physikalisches Institut
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Theoretische Physik

Exercise Sheet 4
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SS 16

Exercises General Relativity and Cosmology

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`http://www.th.physik.uni-bonn.de/klemm/grss16/`

–Homework–

1 Christoffel symbols (15 points)

The metric connection was introduced in the lecture as the unique connection on a manifold M with a given metric g that is

1. torsion-free, i.e. $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$
2. and metric compatible, i.e. $\nabla_\lambda g_{\mu\nu} = 0$.

This will be one of the important objects during your studies of general relativity so it is better to spend some time getting used to it.

1. Use the two properties specifying the metric connection to derive the expression

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\,, \tag{1}$$

for the coefficients of the metric connection. These are also called the *Christoffel symbols*. This shows that the connection satisfying these restrictions is really unique. **3 pts.**

2. On the last sheet you constructed the induced metric on the two-sphere S^2 and the torus T^2 embedded in $\mathbb{R}^3$ as well as de Sitter space embedded in $\mathbb{R}^{1,4}$. They were given by

$$\begin{aligned} ds^2_{S^2} =& R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,,\\ ds^2_{T^2} =& r^2 d\theta^2 + (R + r\cos\theta)^2\, d\phi^2\,,\\ ds^2_{dS^4} =& -dt^2 + \alpha^2\cosh^2(t/\alpha)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]\,. \end{aligned} \tag{2}$$

Calculate the respective Christoffel symbols. **2+2+4 pts.**

3. What is the interpretation of the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$? **2 pts.**

4. Use the Christoffel symbols for the induced metric on T^2 to calculate the Riemann tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\,. \tag{3}$$

What is the Ricci scalar $R = g^{\alpha\beta}R^\rho{}_{\alpha\rho\beta}$? Is the metric flat? **2 pts.**

2 Lagrange formalism with generalized coordinates (7 pts.)

1. Obtain the equations of motion for the general coordinates q^k with metric $g_{ij}(q)$ from the Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \frac{\partial \mathcal{L}}{\partial q^k} = 0\,, \tag{4}$$

   where $\mathcal{L} = \frac{m}{2} g_{ij}(q)\dot{q}^i\dot{q}^j - V(q)$. **3 pts.**

2. Verify that the equations of motion take the form

$$\ddot{q}^l + \Gamma^l_{ij}\dot{q}^i\dot{q}^j = -\frac{1}{m} g^{lk}\frac{\partial V}{\partial q^k}\,. \tag{5}$$

   2 pts.

3. Show that the coefficients Γ^l_{ij} are indeed the Christoffel symbols of the form (1). **2 pts.**

Remark: Note that the variational principle, $\delta\int(g_{ij}\dot{x}^i\dot{x}^j) = 0$ gives the same geodesics as the defining property for geodesics, $\delta\int(g_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}} = 0$, where the derivative is taken with respect to any affine parameter like, for eg, the proper length. This variation gives the Christoffel connection (torsionless and metric-compatible), irrespective of any other connection that may be defined on the manifold. So, in practice, a very fast way of computing Christoffel symbols is to write down the Euler-Lagrange equations for the simplified action and then read off the Christoffel symbols from the resulting geodesic equation.

3 Christoffel symbols for rotating coordinates (6 pts.)

Fictitious forces that are observed in non-inertial frames in Newtonian mechanics can be seen to arise from the metric connection. As an example, let us consider the rotating coordinate system

$$t' = t, \quad x' = (x^2+y^2)^{\frac{1}{2}}\cos(\phi - \omega t), \quad y' = (x^2+y^2)^{\frac{1}{2}}\sin(\phi - \omega t), \quad z' = z, \quad \tan(\phi) = y/x\,. \tag{6}$$

1. Use the results form the previous exercise to calculate the equation of motion for a particle with a flat potential in the non-inertial rotating coordinates given above. **3 pts**

2. Rearrange the result and identify the terms that describe the centrifugal and coriolis forces (both fictitious) that arise in a rotating frame. **3 pts.**