Physikalisches Institut
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Theoretische Physik

Exercise Sheet 5
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SS 16

Exercises General Relativity and Cosmology

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`http://www.th.physik.uni-bonn.de/klemm/grss16/`

–Homework–

1 Satellites in curved space (9 points)

A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1}$$

where

$$\Phi = -\frac{GM}{r} \tag{2}$$

may be thought of as the familiar Newtonian gravitational potential. Here G is Newton's constant and M is the mass of the earth. For this problem Φ may be assumed to be small.

1. Imagine a clock on the surface of the Earth at distance R_1 from the Earth's center, and another clock on a tall building at distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster? **3 pts.**

2. Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$? **3 pts.**

3. How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the earth, neglecting air resistance) completes one orbit? You can work to first order in Φ. Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface? **3 pts.**

2 The Weyl tensor (8 pts.)

The Weyl tensor

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)}Rg_{\rho[\mu}g_{\nu]\sigma} \tag{3}$$

captures the trace-free part of the Riemann tensor.

1. Show that in the vacuum, i.e. for $T_{\mu\nu}g^{\mu\nu} = 0$, the trace of the Riemann tensor vanishes. **1 pts.**

2. Show that the Weyl tensor $C^{\mu}{}_{\nu\rho\sigma}$ is left invariant by a conformal transformation

$$g_{\mu\nu}(x) \to \Omega(x)^2 g_{\mu\nu}(x)\,, \tag{4}$$

where $\Omega(x)$ is an arbitrary but non-vanishing function of spacetime. This can be interpreted as the absence of a characteristic scale under the absence of matter. **3 pts.**

3. Show that for $n \geq 4$ the Weyl tensor satisfies a version of the Bianchi identity,

$$\nabla_{\rho} C^{\rho}{}_{\sigma\mu\nu} = 2\frac{(n-3)}{(n-2)}\left(\nabla_{[\mu} R_{\nu]\sigma} + \frac{1}{2(n-1)} g_{\sigma[\mu}\nabla_{\nu]} R\right)\,. \tag{5}$$

4 pts.

3 The Einstein-Hilbert action (10 pts.)

In the lecture you saw that the Einstein equations can be derived from the variation of the Einstein-Hilbert action

$$S_H = \int d^n x \sqrt{-g} R\,. \tag{6}$$

In this exercise you will fill in some steps that were left out on the blackboard.

1. Using Leibniz rule the variation δS_H can be written as

$$S_H = \int d^n x \left[\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g}\right] = (\delta S)_1 + (\delta S)_2 + (\delta S)_3\,. \tag{7}$$

Express the variation of the Riemann tensor in terms of the variation of the Christoffel symbols. Note that in contrast to the Christoffel symbol itself the variation is actually a tensor. Use the covariant derivative $\nabla_{\lambda}(\delta \Gamma^{\rho}_{\nu\mu})$ to derive

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_{\lambda}(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\lambda\mu})\,. \tag{8}$$

4 pts.

2. Use the above expression to show that $(\delta S)_1$ is the integral over a total divergence. By Stokes theorem this is equivalent to a contribution from the boundary which we assume to vanish. **2 pts.**

3. Use the identity $\mathrm{Tr}(\ln M) = \ln(\det M)$ valid for arbitrary matrices M to derive

$$\delta(g^{-1}) = \frac{1}{g} g_{\mu\nu} \delta g^{\mu\nu}\,. \tag{9}$$

From this derive

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}\,. \tag{10}$$

3 pts.

4. Combine these results to derive the Einstein equations. **1 pts.**