Physikalisches Institut
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Theoretische Physik

Exercise Sheet 6
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SS 16

Exercises General Relativity and Cosmology

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http://www.th.physik.uni-bonn.de/klemm/grss16/

–HOMEWORK–

1 Killing vectors (14 points)

1. Consider the flat Euclidean metric in three dimensions and write down the Killing vector fields corresponding to rotations around the $x-$, $y-$ and $z-$axis. **3 pts.**

2. Transform these vector fields into spherical coordinates to derive the three Killing vector fields for the two-sphere. **3 pts.**

3. Show that they can be labelled R, S, T such that their commutators satisfy the following algebra:

$$\begin{aligned} [R,S] =& T\,, \\ [S,T] =& R\,, \\ [T,R] =& S\,. \end{aligned} \tag{1}$$

   2 pts.

4. Find explicit expressions for a complete set of Killing vector fields for the following spaces:
   a) Minkowski space, with metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$.
   b) A spacetime with coordinates $\{u, v, x, y\}$ and metric

$$ds^2 = -(dudv + dvdu) + a^2(u)dx^2 + b^2(u)dy^2\,, \tag{2}$$

   where a and b are unspecified functions of u. This represents a gravitational wave spacetime. *Hints: There are five Killing vectors in all and all of them have a vanishing u component K^u. Be careful in both cases about the distinction between upper and lower indices.* **2+4 pts.**

2 Christoffel symbols again (10 pts.)

You already got some experience calculating Christoffel symbols. In this exercise you will use a method of calculating them which is often more convenient than plugging in the expression in terms of the metric. Consider curves in a manifold M,

$$\begin{aligned} x:\ (a,b) \subset \mathbb{R} \quad &\to \quad M\,, \\ \sigma \quad &\mapsto \quad x(\sigma)\,, \end{aligned} \tag{3}$$

and define a functional F by

$$F[x] = \frac{1}{2} \int_a^b g_{\mu\nu}(x(\sigma)) \left(\frac{\partial x^\mu(\sigma)}{\partial \sigma} \right) \left(\frac{\partial x^\nu(\sigma)}{\partial \sigma} \right) d\sigma \,. \tag{4}$$

1. Show that the Euler-Lagrange equation for F leads to the geodesic equation

$$\frac{\partial^2 x^\mu}{\partial \sigma^2} + \Gamma^\mu_{\nu\lambda} \left(\frac{\partial x^\nu}{\partial \sigma} \right) \left(\frac{\partial x^\lambda}{\partial \sigma} \right) = 0 \,. \tag{5}$$

   From the geodesic equation one can unambiguously read off the Christoffel symbols. **4 pts.**

2. Explicitly find F for the de Sitter metric introduced on sheet 3 and calculate the Euler-Lagrange equation. From this identify the Christoffel symbols. **6 pts.**