Physikalisches Institut
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Theoretische Physik

Exercise Sheet 7
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SS 16

# Exercises General Relativity and Cosmology

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`http://www.th.physik.uni-bonn.de/klemm/grss16/`

–Homework–

## 1 Metric for the charged black hole (26 points)

In the lecture you have found the Schwarzschild metric which is the vacuum solution to Einstein's equations for a spherically symmetric, static source. Here we generalize this discussion to the case in which the source is charged. It is still static and spherically symmetric, hence we make the same ansatz as for the Schwarzschild solution,

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{1}$$

The non-vanishing components of the Ricci tensor are given by

$$\begin{aligned} R_{tt} =& e^{2(\alpha-\beta)}\left(2\frac{\alpha'}{r} + (\alpha')^2 - \alpha'\beta' + \alpha''\right)\,,\\ R_{rr} =& -(\alpha')^2 + 2\frac{\beta'}{r} + \alpha'\beta' - \alpha''\,,\\ R_{\theta\theta} =& e^{-2\beta}\left(-1 + e^{2\beta} - r\alpha' + r\beta'\right)\,,\\ R_{\phi\phi} =& R_{\theta\theta}\sin^2\theta\,. \end{aligned} \tag{2}$$

Due to the presence of charge we now expect a non-vanishing electromagnetic field strength tensor $F_{\mu\nu}$. We make the ansatz

$$\begin{aligned} F_{tr} =& -F_{rt} = f(r)\,,\\ F_{\theta\phi} =& -F_{\phi\theta} = g(r)\sin\theta\,, \end{aligned} \tag{3}$$

with all other components vanishing.

1. Calculate the energy momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\,, \tag{4}$$

associated to the ansatz we made for the field strength tensor $F_{\mu\nu}$. Demonstrate that this energy momentum tensor is indeed compatible with the spherically symmetric, static ansatz. **(4 pts.)**

2. Solve Maxwell's equations which for a curved spacetime read

$$g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0\,,\, \nabla_{[\mu}F_{\nu\rho]} = 0\,, \tag{5}$$

to obtain

$$g(r) = c_1\,, \quad f(r) = e^{\alpha+\beta}\cdot\frac{c_2}{r^2}\,, \tag{6}$$

with constants $c_1$ and $c_2$. **(4 pts.)**

3. Show that the energy momentum tensor calculated above is traceless. We thus have to solve

$$R_{\mu\nu} = \kappa T_{\mu\nu}\,. \tag{7}$$

**(3 pts.)**

4. Take a suitable linear combination of the $tt$ and $rr$ component of (7) to derive the relation

$$e^{2(\alpha+\beta)} = 1\,. \tag{8}$$

*Note: Strictly speaking you will find* $e^{2(\alpha+\beta)} = const.$ *As in the lecture, this constant can be rescaled to* 1 *without loss of generality.* **(3 pts.)**

5. Consider the remaining equations to find the differential equations

$$\begin{aligned} 2e^{2\alpha}\left(\alpha'' + 2(\alpha')^2 + \frac{2}{r}\alpha'\right) - \kappa f^2 - \kappa\frac{g^2}{r^4} &= 0\,, \\ e^{2\alpha}\left(2r\alpha' + 1\right) - 1 + \frac{\kappa}{2r^2}(g^2 + f^2 r^4) &= 0\,. \end{aligned} \tag{9}$$

**(4 pts.)**

6. Solve the differential equations (9) to find the Reissner-Nordström metric

$$\begin{aligned} ds^2 =& -\Delta dt^2 + \Delta^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad \text{with} \\ \Delta =& 1 - \frac{c_3}{r} + \frac{\kappa}{2}\cdot\frac{c_1^2 + c_2^2}{r^2}\,, \end{aligned} \tag{10}$$

with another constant $c_3$. *Hint: The easy way would be to plug in the ansatz and determine* $c_1$, $c_2$ *and* $c_3$. *For another small simplification you can introduce* $A = e^{2\alpha}$ *and rewrite the differential equations in terms of* $A$. **(4 pts.)**

7. Relate the integration constants $c_1$, $c_2$ and $c_3$ to the mass, electric charge and magnetic charge of the black hole. To this end analyse the asymptotics of the Reissner-Nordström metric and the electromagnetic field $F_{\mu\nu}$. Note that the radial component of the magnetic field is given by $B^r = \epsilon^{01\mu\nu}F_{\mu\nu}$, with $\epsilon^{\rho\sigma\mu\nu} = \frac{1}{\sqrt{-g}}\tilde{\epsilon}^{\rho\sigma\mu\nu}$. For a magnetic monopole of charge $p$ the radial component behaves as $B^r(r) = M/r^2$. **(4 pts.)**