Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 9
17.6.2016
SS 16

# Exercises General Relativity and Cosmology

Prof. Dr. Albrecht Klemm, Thorsten Schimannek

Hand in: 24.6.2016

http://www.th.physik.uni-bonn.de/klemm/grss16/

–Homework–

## 1 Robertson-Walker universe and the Friedmann equations (16 pts.)

In order to describe the evolution of the universe as a whole, we consider the Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \right] . \tag{1}$$

1. Which assumptions where made about the structure of the universe in choosing this metric? **(1 pt.)**

We model the matter and energy content of the universe by a perfect fluid that is comoving with respect to the coordinate system $(t, r, \theta, \phi)$, hence the energy momentum tensor can be written as

$$T_{=} [\rho(t) + p(t)] \, dt^2 + p(t) \cdot g \, . \tag{2}$$

The non-vanishing Christoffel symbols of the metric (1) are

$$\begin{aligned} \Gamma^0_{11} =& \frac{a\dot{a}}{1-\kappa r^2} \, , \quad \Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2} \, , \\ \Gamma^0_{22} =& a\dot{a}r^2 \, , \quad \Gamma^0_{33} = a\dot{a}r^2 \sin^2(\theta) \, , \\ \Gamma^1_{01} =& \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a} \, , \quad \Gamma^1_{33} = \sin^2(\theta)\Gamma^1_{22} = -r(1-\kappa r^2)\sin^2(\theta) \, , \\ \Gamma^2_{12} =& \Gamma^3_{13} = \frac{1}{r} \, , \quad \Gamma^2_{33} = -\sin^2(\theta)\Gamma^3_{23} = -\sin(\theta)\cos(\theta) \, , \end{aligned} \tag{3}$$

and those related to these by symmetry in the lower indices.

2. Show that energy-momentum conservation, most conveniently calculated in the form $\nabla_\mu T^\mu{}_\nu = 0$, is equivalent to

$$\partial_0 \rho = -3\frac{\dot{a}}{a}(\rho + p) \, . \tag{4}$$

Bring this to the form

$$\partial_0(\rho \cdot a^3) = -p\partial_0 a^3 \, , \tag{5}$$

and interpret this in terms of an equation familiar from thermodynamics. **(4 pts)**

3. Show that the off-diagonal elements of the Ricci tensor are zero. For most of them this easily follows from a symmetry argument. Further show

$$R_{00} = -3\frac{\ddot{a}}{a}\,, \quad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1 - \kappa r^2}\,, \quad R_{22} = \frac{R_{33}}{\sin^2(\theta)} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\,. \tag{6}$$

**(5 pts.)**

4. Show that the Ricci scalar is

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]\,. \tag{7}$$

**(2 pts.)**

5. Now consider Einstein's equation in its trace-reversed form (i.e. it only contains the Ricci tensor as well as the energy-momentum tensor and the trace of the energy-momentum tensor). Show that its $\mu\nu = 00$ component gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{8}$$

Use this equation to show that the $\mu\nu = ij$ components give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}\,. \tag{9}$$

Why is there only one distinct equation from $\mu\nu = ij$? Together these equations are known as the Friedmann equations. **(4 pts.)**