Modularity, Periods and Quasiperiods at Special Points in Calabi-Yau Moduli Spaces

Kilian Bönisch

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I hereby declare that this thesis was formulated by myself and that no sources or tools other than those cited were used.

Bonn, ...
Date Signature

1. Gutachter: Prof. Dr. Albrecht Klemm
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Contents

1 Introduction

In this thesis, we investigate the modularity of rank two attractor varieties X. Physically, these manifolds are associated with special black hole solutions of type IIB supergravity. These black holes have non-vanishing charges which means that the intersection

$$H^3(X, \mathbb{Z}) \cap \left(H^{3,0}(X) \oplus H^{0,3}(X)\right)$$

whose elements represent charge vectors must be non-trivial. For rank two attractor varieties, this intersection is 2-dimensional. These varieties are rare and an important question is how one can find them. A very beautiful method for doing this was proposed by Candelas et al. (2019) and is based on arithmetic properties of Calabi-Yau manifolds. For this method, one looks at the local zeta function of X, which can be obtained by reducing X to finite fields. It is conjectured that if X is a rank two attractor variety with 1-dimensional complex structure deformation space, the numerator of this local zeta function factorises to

$$(1 - a_p pT + p^3T^2)(1 - b_pT + p^3T^2).$$

Using this conjectured factorisation allows finding rank two attractor points in the complex structure moduli space for given families of one-parameter Calabi-Yau threefolds. Having calculated the a_p and b_p for some primes p further allows associating automorphic forms to the variety. In the simplest cases, these automorphic forms are just modular forms.

To modular forms one can associate periods and quasiperiods, which then give rise to transcendental numbers. For elliptic curves, the periods of the holomorphic form are rationally related to the periods and quasiperiods of the associated modular form. Klemm, Scheidegger, and Zagier (2020) give numerical evidence that this also happens for rigid Calabi-Yau threefolds. The goal of this thesis is to extend this to rank two attractor varieties. This means in practice that we associate modular forms to rank two attractor varieties and then compare the periods of the holomorphic form and their derivatives with the periods and quasiperiods of the associated modular forms.

Following this introduction, the second chapter gives a short overview of the geometry of Calabi-Yau manifolds and explains their physical relevance. In particular, this contains the definition of periods, which will be the most important tool for us. We conclude by discussing the attractor mechanism. The third chapter introduces modular forms and explains how these give rise to period polynomials. We also show how one can associate L-functions to certain modular forms. In the fourth chapter we discuss the reduction of algebraic varieties to finite fields. We show how such reductions give rise to local zeta functions, which have remarkable properties according to the Weil conjectures. We explain the deformation method which uses the periods for the calculation of local zeta functions. Using the local zeta functions, we define the Hasse-Weil zeta function and discuss cases where it is related to modular forms. We consider the case of elliptic curves in the Legendre family as a first example in the fifth chapter. This chapter serves as an introduction and hence the calculations are rather explicit. For elliptic curves, all the presented aspects are proven and the goal of the following chapters is to generalise this to Calabi-Yau threefolds. In the sixth chapter, we consider our main example, two similar families of Calabi-Yau threefolds which have three rank two attractor points. We explain how these points were found by Candelas et al. (2019) and show that the periods at these points are given by periods and quasiperiods of the associated modular forms. We also use the periods to identify a modular form at a K-Point. In the subsequent chapter, we are concerned with special families of Calabi-Yau threefolds known as self Hadamard products. The considered cases all feature a rank two attractor point at which the deformation method for the calculation of local zeta functions does not work due to a singularity of the Picard-Fuchs equation describing the periods. For these examples, we go in the opposite direction and use the numerical values of the periods to calculate the local zeta functions. We conclude by summarising our results and pointing out directions for future research.

2 Geometry of Calabi-Yau Manifolds

Calabi-Yau manifolds are important geometries for the modeling of spacetime in string theory. Moreover, their mathematical properties are interesting in themselves, and the collaboration of physicists and mathematicians has been very fruitful for the development of the theory of Calabi-Yau manifolds. In this chapter, we give a short introduction to the geometry of Calabi-Yau manifolds. In particular, we explain that they come in families and define the notion of periods as our primary tool for studying the complex structure deformations within these families. We conclude with the discussion of attractor points, which are special points in the complex structure moduli space that are related to black hole solutions in type IIB supergravity.

2.1 Basic Properties of Calabi-Yau Manifolds

This section gives a short physical motivation for considering Calabi-Yau manifolds and briefly discusses their geometry. Looking at deformations of Calabi-Yau manifolds, we see that these can be separated into complex structure deformations and Kähler deformations. In this context, we discuss the mirror symmetry conjecture, too. We conclude by sketching different methods to construct Calabi-Yau manifolds.

Physical Motivation

The physical motivation for considering Calabi-Yau manifolds comes from string theory - a theory of quantum gravity. In contrast to theories based on particles, the basic objects in string theory are strings moving through spacetime. These are described by maps

$$\phi : \Sigma \to M$$

where the 2-dimensional worldsheet Σ and the spacetime M are both Lorentzian manifolds. To describe a realistic low energy particle spectrum, one is lead to superstring theory, where one considers bosonic and fermionic strings whose dynamics are governed by a supersymmetric action. A tremendous simplification of the classical theory comes from the fact that this action is also conformally invariant. However, remaining conformal invariance in the quantized superstring theory requires spacetime to be 10-dimensional. To be able to explain our 4-dimensional macroscopic spacetime, one can make the ansatz that at least locally the spacetime is of the form

$$M = M_4 \times X$$

with the Minkowski space M_4 and X being a compact Riemannian manifold, which in a suitable sense should be small. In the simplest scenario, the metric on M is induced by metrics on M_4 and X. A strength of this setup is that the low energy particle spectrum is determined by the geometry of X in the spirit of the Kaluza-Klein compactification. Candelas et al. (1988) have shown that some remaining unbroken supersymmetry in this spectrum requires the existence of a covariantly constant spinor in a spinor bundle over X. This can be used to define an integrable complex structure J on X, which gives it the structure of a complex manifold. The metric g on X is of type $(1,1)$ concerning this complex structure and thus (X, J, g) is a compact Kähler manifold. Additionally, the existence of the covariantly constant spinor implies that g must be Ricci-flat. In summary, unbroken supersymmetry implies that (X, J, g) is a compact 3-dimensional Ricci-flat Kähler manifold. This directly leads to the definition of Calabi-Yau manifolds. For more information about the physical background, we refer to Becker, Becker, and Schwarz (2006).

Definitions of Calabi-Yau Manifolds

We have seen that physically we are interested in 3-dimensional Ricci-flat Kähler manifolds X. These conditions imply the vanishing of the first Chern class $c_1(TX)$. Calabi (1957) conjectured that for any n-dimensional compact Kähler manifold (X, J, ω) with vanishing first Chern class there exists a unique Ricci-flat Kähler metric g such that the induced Kähler form ω_g is in the same cohomology class as ω. This was later proven by Yau (1977) and motivates our definition of an n-dimensional Calabi-Yau manifold as a simply connected n-dimensional compact Kähler manifold (X, J, ω) with vanishing first Chern class.

Looking at the infinitesimal holonomy of the Levi-Civita connection induced by the Ricci-flat Kähler metric g and using the simple connectedness, one can see that the global holonomy group of n-dimensional Calabi-Yau manifolds is contained in $SU(n)$. This gives a second equivalent definition of Calabi-Yau manifolds. Sometimes Calabi-Yau manifolds are also defined by the stronger requirement that the holonomy group induced by the Ricci-flat Kähler metric is equal to $SU(n)$. In this thesis, we will exclusively consider these Calabi-Yau manifolds. Following Candelas (1988), one can show that in this case the Hodge numbers $h^{i,0}$ vanish for $0 < i < n$. Physically, these Calabi-Yau manifolds give minimal unbroken supersymmetry.

The last equivalent definition which we use is that an n-dimensional Calabi-Yau manifold is a simply connected n-dimensional compact Kähler manifold with nowhere vanishing holomorphic $(n, 0)$-form Ω. This form generates $H^{n,0}(X)$ and we will refer to Ω as the holomorphic form since it is unique up to a rescaling. Candelas (1988) delivers a proof for this and the equivalence to the other definitions.

Deformation of Calabi-Yau Manifolds

We are interested not only in a fixed Calabi-Yau manifold (X, J, ω), but families of them obtained by deformation. Note that we do not regard Calabi-Yau manifolds obtained by holomorphic coordinate changes, i.e.

$$(X', J', \omega') = (\phi(X), (\phi^{-1})^*(J), (\phi^{-1})^*(\omega))$$

with diffeomorphisms $\phi : X \to X'$, as distinct Calabi-Yau manifolds. The remaining deformations can then locally be separated into deformations of the Kähler class, where

$$(X', J') \cong (X, J)$$

as complex manifolds and deformations of the complex structure, where

$$(X', \omega') \cong (X, \omega)$$

as symplectic manifolds. The local structure of the moduli spaces parametrising these deformations is well understood, and an overview can be found in Candelas and de la Ossa (1991). They show that the moduli space locally is indeed a product

$$\mathcal{M}_{\mathrm{cs}} \times \mathcal{M}_{\mathrm{k}}$$

of the moduli spaces of the complex structure and the Kähler form. Locally, these moduli spaces are complex (real) manifolds of dimension

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_{\mathrm{cs}} &= h^{n-1,1} \\ \dim_{\mathbb{R}} \mathcal{M}_{\mathrm{k}} &= h^{1,1}. \end{aligned}$$

Following Tian (1987) and Todorov (1989), the complex structure deformation space of Calabi-Yau manifolds is smooth and unobstructed, i.e. each infinitesimal complex structure lifts to a global one. In this thesis, we will only consider complex structure deformations. The two types of deformations are related by mirror symmetry.

Mirror Symmetry

For the discussion of mirror symmetry, we enlarge the structure of Calabi-Yau manifolds by a B-field[1]

$$B \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$$

which, together with the Kähler form, gives the complexified Kähler form

$$B + i\omega \in H^2(X,\mathbb{C})/H^2(X,\mathbb{Z}).$$

The moduli space then locally decomposes as

$$\mathcal{M}_{\text{cs}} \times \mathcal{M}_{\text{ck}}$$

where the moduli space of the complexified Kähler form is now also a complex manifold of dimension

$$\dim_{\mathbb{C}} \mathcal{M}_{\text{ck}} = h^{1,1}.$$

For a non-rigid family M of Calabi-Yau manifolds, i.e. a family with $h^{1,1} \neq 0$, the statement of the mirror symmetry conjecture is that there exists a mirror family W such that the moduli spaces are isomorphic, but the role of Kähler deformations and complex structure deformations are exchanged

$$\mathcal{M}_{\text{cs}}(M) \cong \mathcal{M}_{\text{ck}}(W)$$
$$\mathcal{M}_{\text{ck}}(M) \cong \mathcal{M}_{\text{cs}}(W).$$

This implies that the Hodge numbers $h^{1,1}$ and $h^{n-1,1}$ of the families are exchanged.

Physically, mirror symmetry relates equivalent theories defined on mirror pairs of Calabi-Yau manifolds. For odd dimension n, this means that IIA string theory on M is equivalent to IIB string theory on W. This makes mirror symmetry an important tool because certain calculations can be easier for one of the two theories and mirror symmetry then allows to relate these results to the other side. This e.g. allowed a famous computation of the numbers of rational curves on the quintic threefold (see Candelas et al. (1991)).

Construction of Calabi-Yau Manifolds

For $n > 1$, there are no known explicit representations of the metric g for Calabi-Yau manifolds with holonomy group $SU(n)$. Making use of Yau's theorem, the idea for the construction of Calabi-Yau manifolds is to construct simply connected compact Kähler manifolds with vanishing first Chern class. If one wants the holonomy group to be $SU(n)$, this has to be checked in addition. In the following we describe well-known methods of constructing Calabi-Yau manifolds. We refer to Joyce (2001) for a more detailed treatment.

In algebraic geometry, results like the adjunction formula allow generating many families of Calabi-Yau manifolds. For example, complete intersection Calabi-Yau manifolds are n-dimensional submanifolds of products of projective spaces

$$\mathbb{CP}^{n_1} \times ... \times \mathbb{CP}^{n_k}$$

of total dimension $N + n$ defined as the vanishing set of N homogeneous polynomials p_i. If this vanishing set does not have singularities and the degrees add up to

$$\sum_{i=1}^{N} \deg p_i = N + n + 1,$$

the first Chern class vanishes, and, by choosing a Kähler form, one has constructed a Calabi-Yau manifold. This method can be generalised for weighted projective spaces. Further generalisations

[1] The B-field can be added to the worldsheet action as a background field to which the strings couple.

are Calabi-Yau manifolds obtained as hypersurfaces in toric varieties. The Calabi-Yau manifolds are then also described by combinatorical data and this description can simplify the construction of the mirror manifolds.

There is also a notion of singular Calabi-Yau manifolds. For these, one exchanges the vanishing of the first Chern class by the triviality of the canonical bundle. This is an equivalent condition for smooth manifolds and also makes sense for certain singular manifolds. Having a singular Calabi-Yau manifold, there may exist a crepant resolution, i.e. one which does not affect the canonical bundle. Then one can use this to get a smooth Calabi-Yau manifold. This is also used to construct smooth Calabi-Yau manifolds out of singular quotients of smooth Calabi-Yau manifolds by a discrete group.

For the construction and the study of Calabi-Yau manifolds, there are powerful physical methods. Witten (1993) introduced techniques by studying 2-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauged linear sigma models which have Calabi-Yau manifolds as moduli space of their vacua. For a recent realisation with abelian and non-abelian gauge groups, we refer to Gerhardus, Jockers, and Ninad (2018).

2.2 Periods of Calabi-Yau Manifolds

We focus on complex structure deformations of Calabi-Yau manifolds. We also restrict to the case of one-parameter models, i.e. Calabi-Yau manifolds with $h^{n-1,1} = 1$. This is done for simplicity, and everything we present in this section can be generalised to the case of higher-dimensional moduli spaces. Throughout this section, we denote by

$$X \to \mathcal{M}$$

a fixed family of n-dimensional Calabi-Yau manifolds with complex structure moduli space[2]

$$\mathcal{M} = \mathbb{CP}^1 \backslash \Sigma,$$

where Σ is a finite set of values for which the Calabi-Yau becomes singular. In the following, we define the notion of periods, which will be the most important tool to study the family X, and discuss properties of these. For this, we follow Klemm (2018).

Definition of Periods

Based on the geometry of our family of Calabi-Yau manifolds, we can define several interesting bundles. Examples include the bundle of k-cycles $H_k(X,\mathbb{Z})$ with fibre $H_k(X_z,\mathbb{Z})$ over $z \in \mathcal{M}$ or the bundle of (p,q)-forms $H^{p,q}(X)$ defined analogously. The change of the fibres in these spaces by going along paths in the base space $\mathcal{M}$ is also called the variation of Hodge structures and is what we are interested in in the following. We will do so by studying the periods of our family.

The main player for the definition of periods is the middle (co)homology of our family X. Having a connection on the bundle $H_n(X,\mathbb{Z})$ allows to parallel transport cycles along paths in the base space. Due to the purely topological nature of the fibres, the parallel transport does not depend on the chosen connection, and the homology for a closed path $\mathcal{C}$ in the base space depends only on the generated homotopy class $[\mathcal{C}] \in \pi_1(\mathcal{M})$. The holomorphic form for our family is a section

$$\Omega \in \Gamma(H^{n,0}(X))$$

which is unique up to multiplication by functions of z. To integrate this form, let us fix a basis $\{\gamma_1^{z_0}, ..., \gamma_{b_n}^{z_0}\}$ for $H_n(X_{z_0},\mathbb{Z})$ at some $z_0 \in \mathcal{M}$. If we choose a contractible subset $U \subset \mathcal{M}$ which contains z_0, we can parallely transport this basis to any point in U. This gives us well-defined sections

$$\gamma_i \in \Gamma(H_n(X|_U,\mathbb{Z}))$$

[2] $\mathcal{M}$ sometimes will be a multi-covering of the complex structure moduli space.

with which we define the period vector by

$$\Pi_i(z) := \int_{\gamma_i(z)} \Omega_z.$$

Note that one can, and we will always choose the holomorphic form Ω so that the periods are holomorphic functions of z.

While Π is a well-defined function on the contractible set U, it will in general not be on $\mathcal{M}$. Analytically continuing Π along a closed path in $\mathcal{M}$, we get the monodromy for that path. Since Ω is well-defined globally, we see that the monodromy is a result of the holonomy on $H_n(X, \mathbb{Z})$. This gives many restrictions on the monodromy of the periods. First of all, the holonomy on $H_n(X, \mathbb{Z})$ can only consist of unimodular elements. Further, we have the intersection product

$$\begin{aligned}\Sigma_z : H_n(X_z, \mathbb{Z}) \times H_n(X_z, \mathbb{Z}) &\to \mathbb{Z} \\ (\gamma, \gamma') &\mapsto \int_{X_z} \hat{\gamma} \wedge \hat{\gamma}'\end{aligned}$$

which is preserved under the holonomy. We can thus conclude that the monodromy of the periods is a group homomorphism

$$M : \pi_1(\mathcal{M}) \to Sp(b_n, \mathbb{Z}).$$

We will use the fact that the periods are the solutions of a differential equation for studying them.

The Picard-Fuchs Equation

We have seen that we can define the periods as holomorphic functions over a contractible subset $U \subset \mathcal{M}$. Taking the derivative of a period Π_i with respect to z, we again have a holomorphic function which is an integral of a form in $H^n(X, \mathbb{C})$ over γ_i, too. Defining the periods over several contractible patches allows us to define the Gauß-Manin connection

$$\nabla : \Gamma(H^n(X, \mathbb{C}) \otimes T\mathcal{M}) \to \Gamma(H^n(X, \mathbb{C}))$$

implicitly by

$$\partial_z^k \Pi_i(z) =: \int_{\gamma_i(z)} \nabla_z^k \Omega_z. \tag{2.1}$$

One can show that the Gauß-Manin connection restricts to

$$\nabla : \Gamma(F^k H^n(X, \mathbb{C}) \otimes T\mathcal{M}) \to \Gamma(F^{k+1} H^n(X, \mathbb{C}))$$

with the Hodge filtrations

$$F^k H^n(X, \mathbb{C}) := \bigoplus_{i \geq k} H^{i,n-i}(X).$$

This fact is known as Griffiths transversality and has two useful consequences. On the one hand, it allows to get a basis of $H^n(X, \mathbb{C})$ by taking derivatives of the holomorphic form Ω and using complex conjugation. On the other hand, we know that the set

$$\{\Omega, \nabla_z \Omega, ..., \nabla_z^{b_n} \Omega\}$$

cannot be linear independent locally over functions in z. Recalling the definition (2.1), the periods are solutions of a linear differential equation of order at most b_n. This differential equation is called the Picard-Fuchs equation. The type of the differential equation is very constrained from the fact that its solutions are periods. For example, it is of Fuchsian type, i.e. it is a linear homogeneous ordinary differential equation with holomorphic coefficients and only regular singular points. This means that we can use the Frobenius method to find a solution around each point. We explain how this works in Section A.1. As we have already seen, another consequence of the geometrical origin of the differential equation is that the monodromy group must be contained in $Sp(b_n, \mathbb{Z})$ up to conjugation. The considered operators will further have a point of maximal unipotent monodromy (MUM-point) at $z = 0$, i.e. a point where all local exponents are equal (and the holomorphic form is chosen such that they are all 0 at that point). There is a lot more one can say about the Picard-Fuchs operators for Calabi-Yau manifolds and people have been trying to find conditions satisfied by these to construct them without referring explicitly to a geometry. These operators are called Calabi-Yau operators and for more information we refer to van Straten (2017).

The Integral Symplectic Basis

In practice, we will obtain the periods as solutions of a Picard-Fuchs equation of order b_n

$$\mathcal{L}\Pi = 0$$

where $\mathcal{L}$ is a differential operator of Fuchsian type. Locally, we use the Frobenius method to get b_n linear independent solutions $\{\varpi_1, ..., \varpi_{b_n}\}$. We will then need to find linear combinations of these solutions that correspond to the periods $\{\Pi_1, ..., \Pi_{b_n}\}$ with respect to an integer symplectic basis $\{\gamma_1, ...\gamma_{b_n}\}$ of $H_n(X, \mathbb{Z})$. We choose this basis so that the intersection product has the form

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

Note that this integer symplectic basis can only be unique up to $Sp(b_n, \mathbb{Z})$ transformations. What can help us to find an integer symplectic basis of periods are the monodromy matrices since we already know that these must be in $Sp(b_n, \mathbb{Z})$ with respect to the canonical symplectic matrix defined above. Up to an overall scaling, this can sometimes be sufficient and if there is some point $z \in \mathcal{M}$ around which one can obtain a period by direct integration this will then also fix the constant[3].

Following Klemm (2018), it is believed that for 3-dimensional one-parameter Calabi-Yau manifolds X with a MUM-point, one can use topological invariants of the mirror family $\tilde{X}$ to fix an integer symplectic basis up to an overall scaling. Without loss of generality, let $z = 0$ be the MUM-point. Then the Frobenius basis of solutions of the Picard-Fuchs equation has the form

$$\varpi(z) = \begin{pmatrix} f_0(z) \\ f_0(z)\log(z) + f_1(z) \\ f_0(z)\log^2(z)/2 + f_1(z)\log(z) + f_2(z) \\ f_0(z)\log^3(z)/6 + f_1(z)\log^2(z)/2 + f_2(z)\log(z) + f_3(z) \end{pmatrix}$$

with power series f_i satisfying $f_0(0) = 1$ and $f_1(0) = f_2(0) = f_3(0) = 0$. Let $\{e\}$ be a positive basis of $H^2(\tilde{X}, \mathbb{Z})$. The topological invariants of $\tilde{X}$ we need are the triple intersection

$$\kappa := \int_{\tilde{X}} e \wedge e \wedge e,$$

the evaluation of the second Chern class against e

$$c_2 := \int_{\tilde{X}} c_2(T\tilde{X}) \wedge e,$$

and the Euler number $\chi(\tilde{X})$. With these, we can write an integer symplectic basis of periods up to an overall constant as

$$\Pi \sim \begin{pmatrix} \frac{\chi(\tilde{X})\zeta(3)}{(2\pi i)^3} & \frac{c_2}{24\cdot 2\pi i} & 0 & \frac{\kappa}{(2\pi i)^3} \\ \frac{c_2}{24} & \frac{\sigma}{2\pi i} & \frac{-\kappa}{(2\pi i)^2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 \end{pmatrix} \varpi \tag{2.2}$$

[3]In most cases, this constant is not important since we can rescale the holomorphic form to get rid of it. However, for our arithmetic considerations, this constant will matter since the period integrals should then be defined algebraically.

where ζ is the Riemann zeta function. Here, one can choose

$$\sigma = \frac{\kappa \bmod 2}{2}.$$

This can also be used to find conditions for the topological invariants based on the Picard-Fuchs equation and its monodromy.

Geometry of the Moduli Space

We have defined the periods as local functions on the complex structure moduli space, which are associated with the variation of the Hodge structure of the middle cohomology. Due to the local Torelli theorem, this can also be inverted locally to define coordinates of the complex structure moduli space by using the periods. This means that there locally is a choice of cycles $\{\gamma_0, ..., \gamma_{h^{n-1,1}}\} \subset H_n(X, \mathbb{Z})$ such that the associated periods $\{\Pi_0, ..., \Pi_{h^{n-1,1}}\}$ are projective coordinates of the complex structure moduli space. For Calabi-Yau manifolds of dimension $n < 3$, there is also a global Torelli theorem which implies that the periods can be used to define global coordinates of the complex structure moduli space.

The complex structure moduli space $\mathcal{M}$ can be given the structure of a Kähler manifold by equipping it with the Weil-Petersson metric. This is induced by the Kähler potential defined by

$$e^{-K} := i^{n^2} \int_X \Omega \wedge \overline{\Omega} = i^{n^2} \Pi^T \Sigma \overline{\Pi}.$$

Note that the expression in terms of the periods is well defined globally even though the periods are not.

Another important function on the complex structure moduli space is the Yukawa coupling or n-point function. Using the Gauß-Manin connection, we define this by

$$\begin{aligned} C_{\underbrace{z...z}_{n\text{-times}}} &: \mathcal{M} \to \mathbb{C} \\ z &\mapsto \int_{X_z} \Omega_z \wedge \nabla_z^n \Omega_z = \Pi^T \Sigma \frac{\mathrm{d}^n}{\mathrm{d}z^n} \Pi. \end{aligned}$$

The expression in terms of the periods is again well defined globally. Also, note that Griffiths transversality implies that

$$\Pi^T \Sigma \frac{\mathrm{d}^k}{\mathrm{d}z^k} \Pi = 0$$

for $k < n$. Following Klemm (2018), one can further show that one can choose the holomorphic form and the coordinate z of $\mathcal{M}$ so that the Yukawa coupling is a rational function. We will always work with such a choice. Another way to determine the Yukawa coupling up to a multiplicative constant is to use the Picard-Fuchs equation. We explain this for the case where the Picard-Fuchs operator is of order $n+1$. Writing the Picard-Fuchs operator as

$$\mathcal{L} = \sum_{i=0}^{n+1} A_i \frac{\mathrm{d}^i}{\mathrm{d}z^i}$$

with the normalisation $A_{n+1} = 1$, we use Griffiths transversality to get

$$\begin{aligned} 0 = \Pi^T \Sigma \mathcal{L} \Pi &= \Pi^T \Sigma \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} \Pi + A_n C_{z...z} \\ &= \left(\frac{n+1}{2} \frac{\mathrm{d}}{\mathrm{d}z} + A_n \right) C_{z...z}. \end{aligned}$$

This determines the Yukawa coupling up to a constant. Having the Yukawa coupling, we can determine the intersection matrix with respect to an arbitrary basis $\tilde{\Pi}$ of periods by solving

$$\tilde{\Pi}^T \tilde{\Sigma} \frac{\mathrm{d}^k}{\mathrm{d}z^k} \tilde{\Pi} = \begin{cases} 0 & , k < n \\ C_{z...z} & , k = n. \end{cases}$$

2.3 The Attractor Mechanism

Attractor points are special points in the complex structure moduli space of Calabi-Yau manifolds. We follow Moore (1998) to motivate the physcial interest in these points and consider type IIB supergravity[4] compactified on a family X of Calabi-Yau threefolds with complex structure moduli space $\mathcal{M}$. The vector multiplet scalars

$$z : M_4 \to \mathcal{M}$$

determine the variation of the complex structure over the spacetime. Concretely, the spacetime is modeled as a bundle with fibre $\{x\} \times X_{z(x)}$ over $x \in M_4$. Since $b_1(X) = 0$, there is only one abelian gauge field

$$G \in \Omega^2(M_4) \otimes H^3(X, \mathbb{R}),$$

where it is understood that it takes values in $H^3(X_{z(x)}, \mathbb{R})$ over $x \in M_4$. This field strength is required to be anti-self-dual with respect to the 10-dimensional Hodge star operator induced by the metric on M_4 and the Ricci-flat Kähler metric on X, i.e.

$$G = - \star_{10} G.$$

We are now interested in black hole solutions of the supergravity equations, which are static, spherically symmetric, and asymptotically Minkowski. We further require these solutions to be BPS solutions. Denoting the radial coordinate of the spatial part of M_4 by r, we can infer that the vector multiplet scalars z only depend on r. By the existence of D-branes, one can now show that the total charge must be integral, i.e.

$$\int_{S^2_\infty} \frac{G}{2\pi} = \hat{\gamma}_{z_\infty} \in H^3(X_{z_\infty}, \mathbb{Z}),$$

where S^2_∞ denotes the spatial boundary of the Minkowski space. For the metric on M_4, we make the ansatz

$$\mathrm{d}s^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}(\mathrm{d}\vec{x})^2.$$

Defining

$$\begin{aligned} Z : \mathbb{R}^+ &\to \mathbb{C} \\ r &\mapsto e^{K/2} \int_{\gamma_{z(r)}} \Omega_{z(r)} \end{aligned}$$

with the Kähler potential K, the holomorphic form Ω and the parallel transport $\gamma_{z(r)}$ of the Poincaré dual of $\hat{\gamma}$, the BPS condition leads to the attractor equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} e^{-U} + \frac{|Z|}{r^2} &= 0 \\ \Pi^{2,1}\left(e^{K/2} z' \nabla_z \Omega - i\frac{e^U}{r^2}\frac{Z}{|Z|}\hat{\gamma}\right) &= 0. \end{aligned} \tag{2.3}$$

Here, $\Pi^{2,1}$ denotes the projection on $H^{2,1}(X)$. For the spacetime to be regular, these equations must evolve towards a fixed point z_* for $r \to 0$. For a non-vanishing charge $\hat{\gamma}$, (2.3) implies that $\Pi^{2,1}\hat{\gamma}_{z_*} = 0$. We need a non-trivial intersection

$$H^3(X_{z_*}, \mathbb{Z}) \cap \left(H^{3,0}(X_{z_*}) \oplus H^{0,3}(X_{z_*})\right)$$

for the solution to exist. Points z_* in the complex structure moduli space for which this is true are called attractor points. Depending on the dimension of the intersection, the attractor points are said to have rank one or two. We later discuss how one can find attractor points of rank two and what modularity properties the underlying Calabi-Yau manifolds have.

[4] This theory can be obtained as a low energy limit of type IIB string theory.

3 Modular Forms and Period Polynomials

We will see that at special points in the complex structure moduli space, we can associate modular forms to the underlying Calabi-Yau manifold. In this chapter, we want to explain what modular forms are and how they give rise to period polynomials, which will later turn out to be connected to the periods of the holomorphic form of the Calabi-Yau manifold. Modular forms are a vast subject and we will only introduce aspects which we will use later. For a more general treatment, we refer to Bruinier et al. (2008). In the sections about period polynomials, we very closely follow Klemm, Scheidegger, and Zagier (2020).

3.1 The Modular Group

Modular forms are functions with specific transformation properties under the action of the modular group, and so we would like to review this here. The full modular group is

$$SL(2,\mathbb{Z}) = \{\gamma \in M_2(\mathbb{Z})|\ \det\gamma = 1\}$$

with $M_2(\mathbb{Z})$ being the set of 2×2 matrices with components in $\mathbb{Z}$. We will also consider congruence subgroups $\Gamma \subset SL(2,\mathbb{Z})$ which are subgroups containing

$$\Gamma(N) := \{\gamma \in SL(2,\mathbb{Z})|\ \gamma \equiv \mathbb{1} \bmod N\}$$

for some $N \in \mathbb{N}^*$. Note that these have a finite index in $SL(2,\mathbb{Z})$. The most important examples for our application are the groups

$$\Gamma_0(N) := \{\gamma \in SL(2,\mathbb{Z})|\ c_\gamma \equiv 0 \bmod N\}$$

where we label the components of any $\gamma \in GL(2,\mathbb{R})$ according to

$$\gamma =: \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

For any $\gamma \in GL^+(2,\mathbb{R})$, we define the action on the complex upper half-plane as a Möbius transformation

$$\begin{aligned} \gamma : \mathbb{H} &\to \mathbb{H} \\ \tau &\mapsto \gamma\tau := \frac{a_\gamma\tau + b_\gamma}{c_\gamma\tau + d_\gamma} \end{aligned}$$

and we also extend this action to elements in

$$\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$$

with the usual arithmetic operations on the Riemann sphere. The reason for doing this is because we want to consider functions on the extended upper half-plane

$$\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

whose quotient

$$\Gamma_0(N)\backslash\overline{\mathbb{H}}$$

can be given the structure of a Riemann surface. The resulting space $X_0(N)$ is called the modular curve of level N and the set of cusps is defined to be the finite subset

$$\Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q}).$$

3.2 Holomorphic Modular Forms

The holomorphic modular forms we consider are holomorphic functions

$$f : \mathbb{H} \to \mathbb{C}$$

with specific transformation properties under the congruence subgroups $\Gamma_0(N)$ and growth conditions at the cusps. To make this concrete, we introduce the notion of Dirichlet characters of level N. We say that

$$\chi : \mathbb{Z} \to \mathbb{C}$$

is a Dirichlet character modulo N if it reduces to a group homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$$

and satisfies $\chi(n) = 0$ for $(n, N) \neq 1$. From now on, we implicitly fix the level N and a Dirichlet character χ. Having defined the action of $GL^+(2, \mathbb{R})$ on the upper half-plane, we also define the weight k slash operator acting on any smooth function on the upper half-plane by

$$(f|_k\gamma)(\tau) := \chi(a_\gamma)(c_\gamma\tau + d_\gamma)^{-k} f(\gamma\tau)$$

with $k \in \mathbb{N}$ and $\gamma \in GL^+(2, \mathbb{R})$. Note that the slash operator respects the group structure, i.e.

$$f|_k\gamma_1|_k\gamma_2 = f|_k(\gamma_1\gamma_2).$$

A weakly holomorphic form of level $k \in \mathbb{N}$ under $\Gamma_0(N)$ with Dirichlet character χ modulo N now is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f|_k\gamma = f \ \forall\gamma \in \Gamma_0(N). \tag{3.1}$$

We are speaking of a weakly holomorphic form because we have not imposed any restrictions on the growth of f regarding the cusps $\Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q})$. To refine the definition, we want to introduce the notion of meromorphicity and holomorphicity of f at the cusps. We start with the cusp generated by ∞. For any level N, we have the translation operator

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$$

and the definition (3.1) implies that f is invariant under translations. We say that f is meromorphic at ∞ if it has a q-expansion

$$f = \sum_{n=m}^{\infty} a_n q^n \tag{3.2}$$

with the q-coordinate $q(\tau) := e^{2\pi i\tau}$. We say f is holomorphic at ∞ if $m \geq 0$. Similarly, we want to define the condition of meromorphicity and holomorphicity of f for any cusp. Let $\alpha \in \mathbb{P}^1(\mathbb{Q})$ represent a cusp and choose a $\gamma \in SL(2, \mathbb{Z})$ such that $\gamma(\infty) = \alpha$. We say that f is meromorphic (holomorphic) at the cusp represented by α if $f|_k\gamma$ is meromorphic (holomorphic) at ∞. This definition is independent of the choice of γ and the representative α. This enables us to finally define the vector space $M_k(\Gamma_0(N), \chi)$ of weight k modular forms under $\Gamma_0(N)$ with Dirichlet character χ modulo N as the space of weakly modular of weight k under Γ which are holomorphic at all cusps. These spaces are finite-dimensional and only non-empty for positive k. A very important subspace will also be the space of cusp forms

$$S_k(\Gamma_0(N), \chi) := \{f \in M_k(\Gamma_0(N), \chi)|\ f \text{ vanishing at all cusps}\}$$

where the vanishing at ∞ means that $m > 0$ in (3.2) and analogously for the other cusps. For the trivial character, we abbreviate the spaces of modular forms by leaving out the character.

Note that for any proper divisor M of N and $D|(N/M)$ we have a map

$$\begin{aligned} S_k(\Gamma_0(M), \chi) &\to S_k(\Gamma_0(N), \chi) \\ f &\mapsto f_D \end{aligned}$$

defined by $f_D(\tau) = f(D\tau)$. Modular forms which arise in that way are called old forms.

3.3 Operators on Holomorphic Modular Forms

Having introduced the spaces $M_k(\Gamma_0(N), \chi)$ of modular forms, we want to define important operators acting on that space and explain how the composition of these operators behaves.

Hecke Operators

Hecke operators are not only very powerful for dealing with the spaces of modular forms but also have direct relevance for modular forms coming from local zeta functions. For fixed level N and character χ, we define Hecke operators

$$T_n : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(N), \chi)$$

for $n \in \mathbb{N}$ with $(n, N) = 1$[1]. To do this we consider the set

$$\mathcal{M}_{n,N} := \{\gamma \in M_2(\mathbb{Z}) |\ \det \gamma = n,\ c_\gamma \equiv 0 \bmod N\}.$$

Any $\gamma \in \Gamma_0(N)$ induces a left action on $\mathcal{M}_{n,N}$, and we can thus consider the quotient $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$. This quotient is finite and in Section A.2 we show that a minimal set of representatives is given by the set

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) |\ ad = n,\ 0 \leq b < d \right\}. \tag{3.3}$$

Note that this set does not depend on the level N, and one can directly see that it has cardinality $\sigma_1(n)$, i.e. the sum of divisors of n. We now define the action of the n'th Hecke operator by

$$f|_k T_n := n^{k-1} \sum_{M \in \Gamma_0(N)\backslash\mathcal{M}_{n,N}} f|_k M.$$

Since f is modular, one can easily see that the sum does not depend on the chosen representatives. The fact that the image of T_n lies in $M_k(\Gamma_0(N), \chi)$ is (up to checking the meromorphicity at the cusps) a consequence of the observation that

$$\gamma^{-1}(\Gamma_0(N)\backslash\mathcal{M}_{n,N})\gamma = \Gamma_0(N)\backslash\mathcal{M}_{n,N}$$

for any $\gamma \in \Gamma_0(N)$. The effect of the Hecke operators on the q-expansion

$$f = \sum_{i=m}^{\infty} a_i q^i$$

is particularly easy. Using the representatives in (3.3), one gets

$$\begin{aligned} f|_k T_n &= \sum_{d|n} \sum_{i=m}^{\infty} \chi(n/d) \frac{n^{k-1}}{d^k} a_i q^{in/d^2} \underbrace{\sum_{b=0}^{d-1} e^{2\pi i b i/d}}_{d\delta_{d|i}} \\ &= \sum_{i=\min(mn,0)}^{\infty} \sum_{r|(n,i)} \chi(r) r^{k-1} a_{in/r^2} q^i. \end{aligned}$$

One can further show that

$$T_m T_n = T_{mn} \quad \text{for} \quad (m, n) = 1$$

and that all Hecke operators can be diagonalised simultaneously on $S_k(\Gamma_0(N), \chi)$. We say that $f \in S_k(\Gamma_0(N), \chi)$ is a Hecke eigenform if it is an eigenvector under the action of all Hecke operators and, additionally, it is normalised so that $a_1 = 1$ in the q-expansion. We then directly see that the eigenvalue of T_n is just a_n. We further define the newspace $S_k^{\text{new}}(\Gamma_0(N), \chi)$ as the space generated by Hecke eigenforms which are not old (as defined in the previous section).

[1]The Hecke operators can be defined for more general n but the case $(n, N) = 1$ will be sufficient for us.

Atkin-Lehner Involutions

To obtain as much computational control as possible over $M_k(\Gamma_0(N), \chi)$, we introduce Atkin-Lehner involutions. For any $Q|N$ with $(Q, N/Q) = 1$, we define the group

$$\mathcal{W}_Q := \frac{1}{\sqrt{Q}} \begin{pmatrix} Q\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & Q\mathbb{Z} \end{pmatrix} \cap SL_2(\mathbb{R}).$$

This group is of interest because it normalises $\Gamma_0(N)$, i.e. for any $W_Q \in \mathcal{W}_Q$ we have

$$W_Q \Gamma_0(N) W_Q^{-1} = \Gamma_0(N)$$

and further, they are involutions in the sense that

$$W_Q^2 \in \Gamma_0(N).$$

We define the action of W_Q on $M_k(\Gamma_0(N), \chi)$ by

$$W_Q : f \mapsto f|_k W_Q$$

and call these Atkin-Lehner involutions. This action does not depend on the chosen element of $\mathcal{W}_Q$ and trivially is an involution. Additionally, one can easily check that for any $f \in M_k(\Gamma_0(N), \chi)$ we have

$$f|_k T_n|_k W_Q = \chi_Q(n) f|_k W_Q|_k T_n$$

where the Dirichlet characters χ_Q modulo Q and $\chi_{N/Q}$ modulo N/Q are uniquely determined by the decomposition

$$\chi = \chi_Q \chi_{N/Q}.$$

In particular, this means that the image of a Hecke eigenform under an Atkin-Lehner involution is again a Hecke eigenform. For a Hecke eigenform $f \in S_k(\Gamma_0(N), \chi)$, there is hence an eigenform $\tilde{f}$ such that

$$f|_k W_Q = \lambda_Q \tilde{f}.$$

We call λ_Q the pseudo-eigenvalue, and Atkin (1978) shows that if f is a newform, then $\tilde{f}$ is also one and λ_Q is algebraic and has absolute value one. For trivial characters, the pseudo-eigenvalue is an ordinary eigenvalue which then must be ± 1.

Complex Conjugation

For $f \in M_k(\Gamma_0(N), \chi)$, we define the complex conjugation by

$$\begin{aligned} M_k(\Gamma_0(N), \chi) &\to M_k(\Gamma_0(N), \overline{\chi}) \\ f &\mapsto \overline{f} \end{aligned}$$

with

$$\overline{f}(\tau) := \overline{f(-\overline{\tau})}.$$

The effect of the complex conjugation on the q-expansion simply corresponds to conjugating the coefficients, and it trivially is an involution. One can easily see that it commutes with the Hecke operators and Atkin-Lehner involutions. Following Atkin (1978), one further finds that for Hecke eigenforms $f \in S_k^{\text{new}}(\Gamma_0(N), \chi)$

$$f|_k W_N = \lambda_N \overline{f}.$$

3.4 *L*-Functions of Modular Forms

To cusp forms $f \in S_k(\Gamma_0(N), \chi)$ with q-expansion

$$f = \sum_{i=1}^{\infty} a_i q^i$$

we can associate a Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Following Bruinier et al. (2008), this converges to a holomorphic function for $\operatorname{Re} s > 1 + \frac{k}{2}$ which can be extended analytically so that $\Gamma(s)L(f,s)$ becomes a holomorphic function on $\mathbb{C}$. If f is a new Hecke eigenform, the multiplicativity of Hecke operators implies that this Dirichlet series has the Euler product

$$L(f,s) = \prod_p (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1}$$

where the product runs over all primes p. One can also show the converse, i.e. if $L(f,s)$ has such an Euler product, then f is a new Hecke eigenform. Further, $L(f,s)$ can be shown to satisfy a functional equation that relates $L(f, k-s)$ with $L(f,s)$. For $f \in S_k^{\text{new}}(\Gamma_0(N), \chi)$, we call $L(f,s)$ the associated L-function. Later we will also introduce L-functions associated with projective algebraic varieties and in some cases this allows us to associate modular forms to them. The Taniyama-Weil conjecture further predicts that L-functions from algebraic varieties always coincide with L-functions of automorphic forms, which are generalised modular forms. For a more detailed discussion of this, we refer to Zagier (2016).

3.5 Period Polynomials

We now want to explain how holomorphic modular forms give rise to period polynomials. An important observation due to Bol (1949) was that for any sufficiently smooth $f : \mathbb{H} \to \mathbb{C}$, we have Bol's identity, i.e. for integers $k \geq 1$, we have

$$D^{k-1}(f|_{2-k}\gamma) = (D^{k-1}f)|_k \gamma \ \forall \gamma \in SL(2, \mathbb{R})$$

with the rescaled derivative $D = \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\tau}$. This can be proven easily by induction on k. We can thus conclude that if f transforms modular with weight $2-k$, then the $(k-1)$'th derivative transforms modular with weight k. Inspired by this, Eichler (1957) studied the converse, namely the modularity properties of the $(k-1)$'th integral of a form of weight k. For this, we consider a fixed $f \in S_k(\Gamma_0(N), \chi)$[2]. We say that $\tilde{f} : \mathbb{H} \to \mathbb{C}$ is an Eichler integral of f if

$$D^{k-1}\tilde{f} = f.$$

We immediately see that the Eichler integral is only unique up to an element in $V_{k-2}^{\mathbb{C}}$, i.e. a polynomial of degree $k-2$ over $\mathbb{C}$. Choosing an Eichler integral $\tilde{f}$, the associated period polynomial $r_{\tilde{f}}$ measures the fail of modularity of $\tilde{f}$, i.e. we define

$$\begin{aligned} r_{\tilde{f}} : \Gamma &\to V_{k-2}^{\mathbb{C}} \\ \gamma &\mapsto \tilde{f}|_{2-k}(\gamma - 1) := \tilde{f}|_{2-k}\gamma - \tilde{f}. \end{aligned}$$

One can easily check that the correction term is a polynomial of degree $k-2$ by using Bol's identity. What we want to understand in the following is the effect that the ambiguity in choosing

[2] Originally, Eichler looked at the case with trivial character. A generalisation which includes non-trivial characters can be found in Gunning (1961).

an Eichler integral has on the period polynomial. For this, it will also be useful to represent the Eichler integral as

$$\tilde{f} = \frac{(2\pi i)^{k-1}}{(k-2)!} \int_{\tau_0}^{\tau} (\tau - z)^{k-2} f(z) \mathrm{d}z,$$

where the ambiguity now lies in the choice of the basepoint $\tau_0 \in \overline{\mathbb{H}}$. The associated period polynomial then is

$$r_{\tilde{f}}(\gamma) = \frac{(2\pi i)^{k-1}}{(k-2)!} \int_{\tau_0}^{\gamma^{-1}\tau_0} (\tau - z)^{k-2} f(z) \mathrm{d}z. \tag{3.4}$$

To understand the period polynomials dependence on the Eichler integral, we look at properties it fulfils. First of all we find that

$$\begin{aligned} r_{\tilde{f}}(\gamma\gamma') &= \tilde{f}|_{2-k}\gamma|_{2-k}\gamma' - \tilde{f} \\ &= \tilde{f}|_{2-k}\gamma|_{2-k}\gamma' - \tilde{f}|_{2-k}\gamma' + \tilde{f}|_{2-k}\gamma' - \tilde{f} \\ &= r_{\tilde{f}}(\gamma)|_{2-k}\gamma' + r_{\tilde{f}}(\gamma'). \end{aligned}$$

This motivates the definition of the group of cocycles

$$Z_k(\Gamma_0(N), \chi) := \{r : \Gamma_0(N) \to V_{k-2}^{\mathbb{C}} |\ r(\gamma\gamma') = r(\gamma)|_{2-k}\gamma' + r(\gamma')\}.$$

It is clear that the ambiguity in choosing the Eichler integral corresponds to elements in the group of coboundaries defined by

$$B_k(\Gamma_0(N), \chi) := \{r : \Gamma_0(N) \to V_{k-2}^{\mathbb{C}} |\ \exists p \in V_{k-2}^{\mathbb{C}} :\ r(\gamma) = p|_{2-k}(\gamma - 1)\}.$$

There is one additional property the period polynomials fulfil. For this, we consider a fixed element $\gamma \in \Gamma_0(N)$ which has a fixed point in $\overline{\mathbb{H}}$. Then there exists a polynomial p such that

$$r_{\tilde{f}}(\gamma) = p|_{2-k}(\gamma - 1).$$

This can be seen most easily by choosing τ_0 to be the fixed point in (3.4). Following Eichler (1957), this additional requirement is only nontrivial in the case where $\mathrm{Tr}(\gamma)^2 = 4$, i.e. when γ is parabolic. One hence defines the parabolic cocycles

$$Z_k^{\mathrm{par}}(\Gamma_0(N), \chi) := \{r \in Z^k(\Gamma_0(N), \chi) |\ \forall \text{ parabolic } \gamma \in \Gamma_0(N) : r(\gamma) \in V_{k-2}^{\mathbb{C}}|_{2-k}(\gamma - 1)\}.$$

Now we know that any $f \in S_k(\Gamma)$ gives a unique class

$$[r_{\tilde{f}}] \in H_k^{\mathrm{par}}(\Gamma_0(N), \chi) := \frac{Z_k^{\mathrm{par}}(\Gamma_0(N), \chi)}{B^k(\Gamma_0(N), \chi)}$$

which does not depend on the chosen Eichler integral. Eichler further shows that the map

$$\begin{aligned} S_k(\Gamma_0(N), \chi) &\to H_k^{\mathrm{par}}(\Gamma_0(N), \chi) \\ f &\mapsto [r_{\tilde{f}}] \end{aligned}$$

together with its complex conjugate[3]

$$\begin{aligned} S_k(\Gamma_0(N), \overline{\chi}) &\to H_k^{\mathrm{par}}(\Gamma_0(N), \chi) \\ f &\mapsto [\overline{r_{\tilde{f}}}] \end{aligned}$$

induce an isomorphism

$$H_k^{\mathrm{par}}(\Gamma_0(N), \chi) \cong S_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \overline{\chi}). \tag{3.5}$$

From now on, we will write $r_{\tilde{f}}$ as r_f and also abbreviate $[r_f]$ as r_f when there is no possible confusion.

[3]The conjugate of a period polynomial is defined by $\overline{r}(\gamma)(\tau) := \overline{r(\gamma)(\overline{\tau})}$.

3.6 Operators on Period Polynomials

We want to use the isomorphism (3.5) to define the action of the operators for holomorphic modular forms on the parabolic cohomology group of period polynomials, too. For this, we consider a fixed group $\Gamma_0(N)$ and character χ.

Hecke Operators

We can define the action of T_n on $H_k^{\text{par}}(\Gamma_0(N), \chi)$ uniquely by demanding that

$$r_f|_{2-k}T_n = r_{f|_kT_n}$$

for all $f \in S_k(\Gamma_0(N), \chi)$ and then extend by linearity. Using Bol's identity, we find that

$$f|_kT_n = n^{k-1}D^{k-1}(\tilde{f}|_{2-k}T_n)$$

where $\tilde{f}$ is an Eichler integral of f and thus $n^{k-1}\tilde{f}|_{2-k}T_n$ is an Eichler integral of $f|_kT_n$. For $\gamma \in \Gamma_0(N)$, we then have

$$r_{f|_kT_n}(\gamma) = \sum_{M \in \Gamma_0(N)\backslash\mathcal{M}_{n,N}} \tilde{f}|_{2-k}M|_{2-k}(\gamma - 1).$$

Now let $M_i \in \Gamma_0(N)\backslash\mathcal{M}_{n,N}$ be the chosen representatives for $i = 1, ..., \sigma_1(n)$. Then

$$M_i\gamma = \gamma_i M_{\pi_\gamma(i)}$$

for some $\gamma_i \in \Gamma_0(N)$ and a permutation π_γ. We can then read of that the action of the Hecke operators on a general $r \in H_k^{\text{par}}(\Gamma_0(N), \chi)$ is given by

$$(r|_{2-k}T_n)(\gamma) = \sum_{i=1}^{\sigma_1(n)} r(\gamma_i)|_{2-k}M_{\pi_\gamma(i)}.$$

Atkin-Lehner Involutions

As in the preceding section, we lift the action of the Atkin-Lehner involutions to the space of period polynomials. For any $f \in S_k(\Gamma_0(N), \chi)$ with an associated Eichler integral $\tilde{f}$, we have by Bol's identity

$$f|_kW_Q = D^{k-1}(\tilde{f}|_{2-k}W_Q)$$

and thus

$$\begin{aligned} r_{f|_kW_Q}(\gamma) &= \tilde{f}|_{2-k}W_Q|_{2-k}(\gamma - 1) \\ &= r_f(W_Q\gamma W_Q^{-1})|_{2-k}W_Q. \end{aligned}$$

Defining the slash operator acting on period polynomials for any normaliser $W \in SL(2, \mathbb{Z})$ of $\Gamma_0(N)$ by

$$(r|_{2-k}W)(\gamma) := r(W\gamma W^{-1})|_{2-k}W,$$

we can write the action of the Aktin-Lehner involutions as

$$r \mapsto r|_{2-k}W_Q.$$

Complex Conjugation

We have already seen that there is an isomorphism

$$H_k^{\mathrm{par}}(\Gamma_0(N),\chi) \cong S_k(\Gamma_0(N),\chi) \oplus S_k(\Gamma_0(N),\overline{\chi}).$$

Now since $f \in S_k(\Gamma_0(N),\chi)$ and $\overline{f} \in S_k(\Gamma_0(N),\overline{\chi})$ have the same Hecke eigenvalues, we are left with a $2_{\mathbb{C}}$-dimensional subspace in $H_k^{\mathrm{par}}(\Gamma_0(N),\chi)$ if we consider all period polynomials with the same Hecke eigenvalues. To separate these, we note that

$$\overline{r}_{\overline{f}} = (-1)^{k+1} r_f|_{2-k}\epsilon$$

where

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is an involution of $\Gamma_0(N)$. This can be seen, e.g. by choosing $\tau_0 = i$ in (3.4). This action trivially commutes with the Hecke operators the Atkin-Lehner involutions and is also an involution. We can use the action

$$r \mapsto (-1)^{k+1} r|_{2-k}\epsilon$$

to decompose each period polynomial into two parts

$$r = r^+ + r^-$$

corresponding to their eigenvalues.

3.7 Meromorphic Modular Forms

We saw that to holomorphic modular forms $f \in S_k(\Gamma_0(N),\chi)$ we can associate unique cohomology classes $[r_f] \in H_k^{\mathrm{par}}(\Gamma_0(N),\chi)$ of period polynomials. Eichler studied not only holomorphic modular forms, which he calls automorphic forms of first kind, but also automorphic forms of the second kind, which are meromorphic modular forms

$$M_k^{\mathrm{mero}}(\Gamma_0(N),\chi) := \{F : \overline{\mathbb{H}} \to \mathbb{C} |\ F \text{ meromorphic and } F|_k\gamma = F\ \forall \gamma \in \Gamma_0(N)\}$$

for which the integral

$$\int_{\tau_0}^{\tau} (\tau - z)^{k-2} F(z) \mathrm{d}z$$

does not depend on the path of integration. This means that F is locally a $(k-1)$'th derivative and we hence define

$$S_k^{\mathrm{mero}}(\Gamma_0(N),\chi) := \{F \in M_k^{\mathrm{mero}}(\Gamma_0(N),\chi) |\ F \text{ locally a } (k-1)\text{'th derivative}\}.$$

One can easily see that elements in $S_k^{\mathrm{mero}}(\Gamma_0(N),\chi)$ are not in 1 to 1 correspondence to cohomology classes of period polynomials since adding elements from

$$D^{k-1} M_{2-k}^{\mathrm{mero}}(\Gamma_0(N),\chi)$$

does not change the period polynomial. We thus define

$$\mathbb{S}_k(\Gamma_0(N),\chi) := S_k^{\mathrm{mero}}(\Gamma_0(N),\chi)/D^{k-1} M_{2-k}^{\mathrm{mero}}(\Gamma_0(N),\chi).$$

This space is finite-dimensional and according to Klemm, Scheidegger, and Zagier (2020) there is an isomorphism

$$S_k(\Gamma_0(N),\chi) \cong \mathbb{S}_k(\Gamma_0(N),\chi)/S_k(\Gamma_0(N),\chi) =: \check{S}_k(\Gamma_0(N),\chi).$$

The Hecke operators, Atkin-Lehner involutions, and the complex conjugation on $\mathbb{S}_k(\Gamma_0(N),\chi)$ are defined exactly as for holomorphic modular forms and satisfy the same relations. In Section A.4, we explain how we find merormorphic forms $F \in \mathbb{S}_k(\Gamma_0(N),\chi)$ with the same Hecke eigenvalues as a given Hecke eigenform $f \in S_k(\Gamma_0(N),\chi)$. Note that we will always choose F so that it only has poles at the cusps.

3.8 Rationality of Period Polynomials and the Legendre Relations

Given a Hecke eigenform $f \in S_k(\Gamma_0(N), \chi)$, we obtain a unique cohomology class in $H_k^{\mathrm{par}}(\Gamma_0(N), \chi)$. If χ is trivial and thus f is well defined on $X_0(N)$, it is stated by Klemm, Scheidegger, and Zagier (2020) that the representative r_f can be chosen such that

$$r_f^{\pm} = \omega^{\pm} r^{\pm}$$

where the polynomials $r^{\pm}(\gamma)$ are defined over $\mathbb{Q}(f)$ for all $\gamma \in \Gamma_0(N)$, i.e. over the field of rational numbers extended by the fourier coefficients of f. How we make this choice is explained in Section A.3. In this way, we obtain the periods $\omega^{\pm}$ of f, which are unique up to factors in $\mathbb{Q}(f)$. To f we can associate a meromorphic form $F \in \mathbb{S}_k(\Gamma_0(N))$ with the same Hecke eigenvalues and the same coefficient field $\mathbb{Q}(f)$. The representatives of the period polynomial can then be chosen such that

$$r_F^{\pm} = \eta^{\pm} r^{\pm}.$$

We call $\eta^{\pm}$ the quasiperiods of f. Note that the quasiperiods of f are only unique up to factors in $\mathbb{Q}(f)$ and shifts by elements of $\omega^{\pm}\mathbb{Q}(f)$ due to the non-uniqueness of F. In summary, we see that a Hecke eigenform f gives rise to four transcendental numbers. These numbers satisfy the so called Legendre-relations. These are most easily stated when F is chosen such that it only has a pole at ∞. Following Klemm, Scheidegger, and Zagier (2020), we then have

$$\frac{1}{(2\pi i)^{k-1}}(\omega^+ \eta^- - \omega^- \eta^+) \in \mathbb{Q}(f).$$

If F features poles not only at ∞ but also at other cusps, then $\mathbb{Q}(f)$ must be replaced by the rationals extended by the coefficients of F expanded around these cusps.

Later we want to generalise this to forms with non-trivial character. To do so, we consider these as forms with trivial character with respect to the subset

$$\Gamma_1(N) := \{\gamma \in \Gamma_0(N) |\ a_\gamma, d_\gamma \equiv 1 \bmod N\}$$

and observe that everything we stated remains true.

4 Arithmetic Aspects of Calabi-Yau Manifolds

Given a Calabi-Yau manifold defined by the vanishing set of homogeneous polynomials in $\mathbb{CP}^N$ for some N, one may also consider the set over some finite field $\mathbb{K}$, i.e. the vanishing set in $\mathbb{KP}^N$. To such a reduction, we can associate a zeta function. This function has remarkable properties and also contains information about the smooth manifold. For special Calabi-Yau manifolds, it further allows us to associate modular forms to the manifolds. In this chapter, we show how this works by mainly following Zagier (2016) and also explain how the periods of Calabi-Yau manifolds can be used to compute the zeta function. We conclude by discussing the application for special Calabi-Yau manifolds.

4.1 Reduction to Finite Fields and the Zeta Functions

Let X be a non-singular projective algebraic variety[1] defined over $\mathbb{Q}$. After a rescaling, we can also take the coefficients in the defining equations to be in $\mathbb{Z}$. The coefficients can now be uniquely injected in $\mathbb{F}_p \subset \mathbb{F}_{p^k}$ which allows us to define the variety $X/\mathbb{F}_{p^k}$ over the finite field $\mathbb{F}_{p^k}$. The reduction to $\mathbb{F}_p$ is smooth[2] for all but finitely many primes p and the reduction to $\mathbb{F}_{p^k}$ will then also be smooth for all k. We then say that p is a prime of good reduction. Note that if the reduction is not smooth, this does not necessarily mean that the prime is a bad prime since there may exist a different model for X which admits a smooth reduction to $\mathbb{F}_p$. Our main object of interest will be the number of points of the manifold over finite fields, i.e. the numbers

$$N_{p^k} := |X/\mathbb{F}_{p^k}|$$

for primes p of good reduction. With these, we define the local zeta function by

$$\zeta(X/\mathbb{F}_p, T) := \exp\left(\sum_{k=1}^{\infty} \frac{N_{p^k}}{k} T^k\right).$$

This is first defined as a formal power series in T. According to the Weil conjectures (see Weil (1949)), which are now proven, this function has remarkable properties and contains information about the topology of the complex manifold X:

- The local zeta function is rational and further has the form

$$\zeta(X/\mathbb{F}_p, T) = \prod_{i=0}^{2n} P_i(X/\mathbb{F}_p, T)^{(-1)^{i+1}}$$

  where the integral polynomials P_i factors over $\mathbb{C}$ as

$$P_i(X/\mathbb{F}_p, T) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} T)$$

  with b_i being the i'th Betti number. We further have

$$P_0(X/\mathbb{F}_p, T) = 1 - T$$
$$P_{2n}(X/\mathbb{F}_p, T) = 1 - p^n T.$$

- All zeros of $P_i(X/\mathbb{F}_p, T)$ have absolute value $p^{-i/2}$, i.e. one has $|\alpha_{ij}| = p^{i/2}$.

[1] This includes all Calabi-Yau manifolds with full $SU(n)$ holonomy for $n \geq 3$ (see Joyce (2001)).
[2] This can locally be checked by looking at the rank of the Jacobian as one can also do for varieties over $\mathbb{R}$ or $\mathbb{C}$.

- Denoting the Euler number of X by χ each P_i satisfies a functional equation

$$P_{2n-i}\left(\frac{1}{p^nT}\right) = \pm_i \frac{P_i(T)}{(p^{n/2}T)^{b_i}}.$$

For the local zeta function, this implies

$$\zeta\left(X/\mathbb{F}_p, \frac{1}{p^nT}\right) = \pm_n \left(p^{n/2}T\right)^{\chi} \zeta\left(X/\mathbb{F}_p, T\right).$$

Following Deligne (1974), we further have that $\pm_n = +$ for odd n.

Note that in the same way we can consider $\zeta(X/\mathbb{F}_q, T)$ with $q = p^k$ for some integer $k > 0$. The Weil conjectures then hold upon replacing p by q.

The fact that we obtained $X/\mathbb{F}_p$ by reducing a variety defined over $\mathbb{Q}$ allows us to define global L-functions

$$L_i(X/\mathbb{Q}, s) = \prod_p P_i(X/\mathbb{F}_p, p^{-s})^{-1}$$

which converge to meromorphic functions for $\operatorname{Re} s \gg 0$. Here, the product runs over all primes p. In particular, there is also a definition of $P_i(X/\mathbb{F}_p, T)$ for bad primes which we omit here. In this sight, we also call $P_i(X/\mathbb{F}_p, T)$ the local L-factors. In the same way, the Hasse-Weil zeta function is defined by

$$\zeta(X/\mathbb{Q}, s) = \prod_p \zeta(X/\mathbb{F}_p, p^{-s})$$

which again converges for $\operatorname{Re} s \gg 0$. It is conjectured that the global L-functions can be analytically continued to meromorphic functions on the complete complex plane. Further, it is believed that L_i satisfies a functional equation relating $L_i(X/\mathbb{Q}, s)$ and $L_i(X/\mathbb{Q}, i+1-s)$.

We already defined L-functions for new Hecke eigenforms, where it was a consequence of the multiplicativity of the Hecke operators that the L-function can be written as an Euler product. It is believed that the global L-functions of projective algebraic varieties defined over $\mathbb{Q}$ always coincide with L-functions of certain automorphic forms[3]. In this thesis, we restrict to special Calabi-Yau manifolds where the Hasse-Weil zeta function gives rise to modular forms.

4.2 Local L-Factors from Periods

The local L-factors can, by definition, be computed by counting the number of points N_{p^k} for finitely many k. However, for higher dimensions of X and bigger primes p this quickly becomes practically infeasible. In this section, we show examples where $L_n(X_z/\mathbb{F}_p, T)$ for n-dimensional families X of Calabi-Yau manifolds can be computed by using only the periods of X. This will allow us to do practical calculations and shows that the periods contain information about the reduction of X to finite fields.

The Legendre Curve

Let X be an elliptic curve defined over $\mathbb{Q}$. The Weil conjectures (together with $\pm_1 = +$) imply that the local zeta function has the form

$$\zeta(X/\mathbb{F}_p, T) = \frac{1 - a_pT + pT^2}{(1-T)(1-pT)}$$

for some $a_p \in \mathbb{Z}$. The Weil conjectures also directly give Hasse's theorem, i.e.

$$|a_p| \leq 2\sqrt{p}. \tag{4.1}$$

[3] Automorphic forms are a generalisation of modular forms.

We now show how the periods of a family of elliptic curves can be used to calculate the zeta function. From a practical point of view, this is not necessary but will serve as a motivation for the case of 3-dimensional Calabi-Yau manifolds. We consider the Legendre family defined by the vanishing sets X_z of the polynomial

$$P_z(X,Y,Z) := Y^2Z - X(X-Z)(X-zZ)$$

in $\mathbb{CP}^2$ for $z \in \mathbb{CP}^1\backslash\{0,1,\infty\}$. In Chapter 5, we consider this family in more detail and show that the holomorphic period around $z=0$ has the form

$$\varpi_0(z) = \sum_{k=0}^{\infty} c_k z^k$$

with

$$c_k = \binom{-1/2}{k}^2.$$

Dwork found a way to calculate a p-adic unit root $r_p(z) \in \mathbb{Z}_p$ by using the periods such that the numerator of the zeta function for primes p of good reduction for X_z is given by

$$P_1(X_z/\mathbb{F}_p, T) = (1 - r_p(z)T)\left(1 - \frac{p}{r_p(z)}T\right).$$

For a review, we refer to Katz (1973). In particular, we must have

$$a_p(z) = r_p(z) + \frac{p}{r_p(z)} \in \mathbb{Z} \subset \mathbb{Z}_p.$$

Dwork proved that this unit root is given by

$$r_p(z) = (-1)^{(p-1)/2}\frac{\varpi_0(t)}{\varpi_0(t^p)}|_{t=\tilde{z}}$$

where $\tilde{z}$ is the Teichmüller lift of $z \in \mathbb{Q}$. The p-adic analytic continuation of $\frac{\varpi_0(t)}{\varpi_0(t^p)}$ can be done easily due to the Dwork congruences discussed, for example, by Mellit and Vlasenko (2016). For primes $p \geq 17$, the Hasse bound (4.1) implies that it is sufficient to calculate $a_p(z) \bmod p$. By the definition of the Teichmüller lift, one has $\tilde{z} \equiv z \bmod p$ and with the congruence

$$c_{rp+s} \equiv c_r c_s \bmod p$$

we conclude that

$$a_p(z) \equiv (-1)^{(p-1)/2} \sum_{k=0}^{p-1} \binom{-1/2}{k}^2 z^k \bmod p.$$

One-Parameter Calabi-Yau Threefolds

Let X be a one-parameter Calabi-Yau threefold defined over $\mathbb{Q}$. By the Weil conjectures (together with $\pm_3 = +$), we know that

$$P_3(X/\mathbb{F}_p, T) = 1 + \alpha_p T + \beta_p p T^2 + \alpha_p p^3 T^3 + p^6 T^4$$

for some $\alpha_p, \beta_p \in \mathbb{Z}$ bounded by

$$|\alpha_p| \leq 4p^{3/2}, \quad -2p^2 \leq \beta_p \leq 6p^2. \tag{4.2}$$

For the case we consider in chapter 6, following Candelas et al. (2019), the complete local zeta function further has the form

$$\zeta(X/\mathbb{F}_p, T) = \frac{P_3(X/\mathbb{F}_p, T)}{(1-T)(1-pT)^{h^{1,1}}(1-p^2T)^{h^{1,1}}(1-p^3T)}.$$

Now let X be a family of one-parameter Calabi-Yau threefolds with complex structure parameter z. Further, X_z should be defined over $\mathbb{Q}$ for $z \in \mathbb{Q}$ and $z = 0$ should be a MUM-point. Samol (2010) and van Straten (2019) conjectured the deformation method which we explain in the following for the computation of $P_3(X_z/\mathbb{F}_p, T)$. One can define cohomology groups to which the Frobenius action

$$\begin{aligned} \mathbb{F}_{p^k} &\to \mathbb{F}_{p^k} \\ x &\mapsto x^p \end{aligned}$$

extends. In our case, this implies that we can write

$$P_3(X_z/\mathbb{F}_p, T) = \det(1 - TF_p(t))|_{t=\tilde{z}}$$

with a 4×4 matrix

$$F_p(t) \in \mathbb{Q}[[t]]^{4\times 4}$$

called the Frobenius and the Teichmüller lift $\tilde{z}$ of z. This Frobenius can be computed in terms of the Frobenius basis of solutions given by

$$\varpi(z) = \begin{pmatrix} f_0(z) \\ f_0(z)\log(z) + f_1(z) \\ f_0(z)\log^2(z)/2 + f_1(z)\log(z) + f_2(z) \\ f_0(z)\log^3(z)/6 + f_1(z)\log^2(z)/2 + f_2(z)\log(z) + f_3(z) \end{pmatrix}$$

with power series f_i satisfying $f_0(0) = 1$ and $f_1(0) = f_2(0) = f_3(0) = 0$. Defining

$$B(z)_{i,j} := \Theta^j \varpi_i(z)|_{\log(z)=0}$$

for $0 \le i, j \le 3$ with $\Theta := z\frac{\mathrm{d}}{\mathrm{d}z}$, the statement is that the Frobenius is given by

$$F_p(z) = B(z^p)^{-1} F_p(0) B(z)$$

with

$$F_p(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ \gamma_p p^3 & 0 & 0 & p^3 \end{pmatrix}$$

and $\gamma_p \in \mathbb{Z}_p$. What makes this method powerful is that the computation of the power series $F_p(z)$ up to a certain p-adic order of the coefficients requires the expansion up to a finite order in z. This is explained by the following observations in the case that the Picard-Fuchs operator does not have apparent singularities:

- $F_p(z)$ is p-adically integral, i.e. $F_p(z) \in \mathbb{Z}_p[[z]]^{4\times 4}$.
- $F_p(z)$ mod p^3 does not depend on γ_p and is polynomial in z.
- For each $s > 3$, there is a unique choice of γ_p mod p^{s-3} such that $F_p(z)$ is rational in z. Moreover, the denominator of this rational function is given by an exponentiation of the discriminant of the Picard-Fuchs operator.

Expanding up to higher p-adic order of the coefficients thus determines a unique $\gamma_p \in \mathbb{Z}_p$. Following Thorne (2018), it was observed that γ_p is up to the ratio of topological invariants of the mirror manifold given by $\zeta_p(3)$, the p-adic zeta function evaluated at 3. It is believed that the appearance of this value is related to that of $\zeta(3)$ in the matrix relating the Frobenius basis of periods to the integer symplectic basis.

Due to the bounds given in (4.2), it is sufficient to calculate $F_p(z)$ mod p^4 for primes $p \ge 11$. In practice, we will hence only determine γ_p mod p and calculate up to this order.

If the Picard-Fuchs operator has apparent singularities, the last statement remains true and allows the computation of $F_p(z)$ for all regular points of the Picard-Fuchs operator. This method fails for apparent singularities and this is why we need a different strategy in Chapter 7.

Rigid Calabi-Yau Threefolds and *K*-Points

Rigid Calabi-Yau threefolds are 3-dimensional Calabi-Yau manifolds with $h^{2,1} = 0$, i.e. they do not admit complex structure deformations. Thus, the deformation method seems not to be applicable. It turns out that this is not correct since the singular fibres of one-parameter families of Calabi-Yau threefolds can give rise to rigid Calabi-Yau manifolds. Following Samol (2010), it is expected that if $z \in \mathbb{Q}$ is a conifold there is a resolution $\hat{X}_z$ of X_z which is a rigid Calabi-Yau manifold. A priori it was not expected that the deformation method gives sensible results for points z corresponding to singular fibres. However, it was observed that it does for conifold singularities. Following Thorne (2018), the Frobenius $F_p(z)$ then has a vanishing eigenvalue and we get the factorisation

$$\det(1 - TF_p(t))|_{t=\tilde{z}} = (1 - p\chi_p T)(1 - a_p T + p^3 T^2)$$

with $\chi_p = \pm 1$. The quadratic factor gives the L-factor of the rigid Calabi-Yau manifold

$$P_3(\hat{X}_z/\mathbb{F}_p, T) = (1 - a_p T + p^3 T^2).$$

By the Weil conjectures, we further have

$$|a_p| \leq 2p^{3/2}.$$

For K-Points, i.e. singularities of the Picard-Fuchs equation with local exponents $(1, 1, 2, 2)$, Thorne (2018) states that the Frobenius $F_p(z)$ has two vanishing eigenvalues which leads to the form

$$\det(1 - TF_p(t))|_{t=\tilde{z}} = (1 - a_p T + p^2 \chi_p T^2)$$

with $\chi_p = \pm 1$. He further says that for typical examples (but not always) a_p gives rise to modular forms of weight 3 for some $\Gamma_0(N)$. Note that these must have a non-trivial character since holomorphic modular forms of odd weight do not exist for trivial characters.

4.3 Modularity of certain Calabi-Yau Manifolds

In this section, we discuss the modularity of elliptic curves and special Calabi-Yau threefolds. The key object for this is the global L-function $L_n(X/\mathbb{Q}, s)$.

Modularity of Elliptic Curves

Let X be an elliptic curve defined over $\mathbb{Q}$. The Weil conjectures (together with $\pm_1 = +$) imply that the local zeta function has the form

$$\zeta(X/\mathbb{F}_p, T) = \frac{1 - a_p T + pT^2}{(1 - T)(1 - pT)}$$

for primes p of good reduction and some $a_p \in \mathbb{Z}$. The Weil conjectures also directly give Hasse's theorem, i.e.

$$|a_p| \leq 2\sqrt{p}.$$

One can see that the L-factors

$$\frac{1}{1 - a_p p^{-s} + p^{-2s+1}}$$

have the form of the L-factors of the L-function of a Hecke eigenform of weight 2 and it was proven by Wiles that there exists a new Hecke eigenform $f \in S_2^{\text{new}}(\Gamma_0(N))$ such that

$$L_1(X/\mathbb{Q}, s) = L(f, s).$$

Here, the level N is the conductor of the elliptic curve X. The converse is also true, i.e. one can find an associated elliptic curve for each Hecke eigenform $f \in S_2^{\mathrm{new}}(\Gamma_0(N))$. The geometric origin for this is that there is a modular parametrisation

$$\phi : X_0(N) \to X$$

mapping $\infty \in X_0(N)$ to the identity of X and such that the pullback of the canonical one-form of X is up to a rational factor given by $2\pi i f \mathrm{d}\tau$. In particular, it follows that the values of the periods of the elliptic curve and their derivatives are rationally related to the periods and quasiperiods of f.

Rigid Calabi-Yau Threefolds

The first case we want to consider are rigid Calabi-Yau threefolds defined over $\mathbb{Q}$. By the Weil conjectures (together with $\pm_3 = +$), we then have

$$P_3(X/\mathbb{F}_p, T) = 1 - a_p T + p^3 T^2$$

for some $a_p \in \mathbb{Z}$ satisfying

$$|a_p| \leq 2p^{3/2}.$$

We see that the global L-function $L_3(X/\mathbb{Q}, s)$ has the form of a L-function of a weight 4 modular form and Gouvêa and Yui (2011) indeed show that

$$L_3(X/\mathbb{Q}, s) = L(f, s)$$

for some $f \in S_4^{\mathrm{new}}(\Gamma_0(N))$. However, this relation is not as well studied as for elliptic curves. For example, one does not know which levels N one can get from rigid Calabi-Yau threefolds. Following Candelas et al. (2019), it is believed that the geometric origin of this is a correspondence between X and the Kuga-Sato threefold $\mathcal{E}^{(2)}$ which is the 2-fold fibre product of the elliptic modular surface[4] $\mathcal{E} \to X_0(N)$. Cusp forms of weight k under $\Gamma_0(N)$ can be identified with $(k-1)$-forms on $\mathcal{E}^{(k-2)}$ and the modular form arises from pulling back the holomorphic form on X. The periods and quasiperiods of f are the periods of 3-forms on $\mathcal{E}^{(2)}$ and we hence expect to be able to express the periods of the holomorphic form of X in terms of these.

Rank Two Attractor Varieties

In the following, we will consider the case where X is an attractor variety of rank two, i.e. we have a splitting

$$H^3(X, \mathbb{Q}) = \Lambda \oplus \Lambda_\perp$$

for

$$\begin{aligned} \Lambda &\subset H^{3,0}(X) \oplus H^{0,3}(X) \\ \Lambda_\perp &\subset H^{2,1}(X) \oplus H^{1,2}(X). \end{aligned}$$

Candelas et al. (2019) show (based on the Hodge conjecture) that this implies the factorisation

$$P_3(X/\mathbb{F}_p, T) = (1 - a_p(pT) + p(pT)^2)(1 - b_p T + p^3 T^2)$$

for $a_p, b_p \in \mathbb{Z}$ and the Weil conjectures give the bounds

$$|a_p| \leq 2p\ , \quad |b_p| \leq 2p^{3/2}.$$

[4]This is a special bundle which has elliptic curves as generic fibres.

Writing the factorisation in this form suggests that a_p can be related to a weight 2 modular form while b_p can be related to a weight 4 modular form. This was proven by Gouvêa and Yui (2011), and attractor varieties of rank two defined over $\mathbb{Q}$ give rise to modular forms

$$\begin{aligned} f_2 &\in S_2(\Gamma_0(N)) \\ f_4 &\in S_4(\Gamma_0(N)). \end{aligned}$$

On the level of the Hodge structure, the attractor variety can be understood as a mix of a rigid Calabi-Yau threefold with an elliptic curve and this is also visible in the global L-function L_3. As for the case of rigid Calabi-Yau threefolds, the geometric origin should be a correspondence between the Kuga-Sato threefold and X. The weight 4-form then arises also as it does for rigid Calabi-Yau threefolds.

Following Candelas et al. (2019), it is further believed that there is a surface S sitting in the diagram

$$\begin{array}{ccccc} X_0(N) \times X & \supset & S & \longrightarrow & X \\ & & \downarrow & & \\ & & X_0(N) & & \end{array}$$

with the natural projections such that the weight 2 form comes from pulling back the $(2,1)$-part of $\nabla_z \Omega$ and integrating over the fibres over $X_0(N)$. In total, we expect to be able to express the periods of the holomorphic form and all its derivatives in terms of the periods and quasiperiods of f_2 and f_4.

5 The Legendre Family

From the physical point of view, our main interest lies in studying families of 3-dimensional Calabi-Yau manifolds. However, to make things simpler, we will start with the case of elliptic curves as 1-dimensional Calabi-Yau manifolds[1]. The family of elliptic curves we want to consider is the Legendre family. This is defined by the vanishing set X_z of the polynomial

$$P_z(X,Y,Z) := Y^2Z - X(X-Z)(X-zZ)$$

in $\mathbb{CP}^2$. Here, $z \in \mathbb{CP}^1$ is complex and parametrises the complex structure of the curve[2]. Calculating the Jacobian, one finds that the set of singular points is given by

$$\Sigma = \{0, 1, \infty\}.$$

5.1 The Picard-Fuchs Equation and its Solutions

Following Section A.6, we can represent period integrals as

$$\Pi(z) = \int_{T_z(\gamma(z))} \frac{1}{P_z}\mu.$$

Since $h^{2,1} = 1$, we expect the Picard-Fuchs equation to be of second order. We hence take the second derivative of the period and using SageMath to decompose polynomials into ideals, we find that

$$\begin{aligned}
\partial_z^2\Pi(z) &= \int_{T(\gamma(z))} \frac{2(XZ^2 - X^2Z)^2}{P_z^3}\mu \\
&= \frac{1}{1+z}\int_{T(\gamma(z))} \frac{-\frac{4}{3}XZ^3\partial_X P_z + (X^2YZ - \frac{1}{3}YZ^3)\partial_Y P_z + (-2X^2Z^2 + \frac{2}{3}Z^4)\partial_Z P_z}{P_z^3}\mu \\
&= \frac{1}{1+z}\int_{T(\gamma(z))} \frac{\frac{1}{2}Z^3 - \frac{3}{2}X^2Z}{P_z^2}\mu \\
&= \frac{1}{1+z}\int_{T(\gamma(z))} \frac{\frac{1}{2z}Z\partial_X P_z + \frac{1+z}{4z(z-1)}Y\partial_Y P_z - \frac{1+z}{2z(z-1)}Z\partial_Z P_z + \frac{1-z-2z^2}{z(z-1)}(X^2Z - XZ^2)}{P_z^2}\mu \\
&= -\frac{1}{4}\frac{1}{z(z-1)}\Pi(z) + \frac{1-z-2z^2}{z(z-1)(z+1)}\partial_z\Pi(t)
\end{aligned}$$

and so the Picard-Fuchs equation is given by $\mathcal{L}\Pi = 0$ with the differential operator

$$\mathcal{L} = 4(z-1)\Theta^2 + 4z\Theta + z$$

and $\Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$. As expected, $\mathcal{L}$ has only regular singularities. These coincide with the set Σ where X_z becomes singular. Note that this does not have to be the case in general. Calculating the indicial equation for each singular point, we find that the Riemann symbol is

$$\mathcal{P}\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{array}\right\}.$$

[1] We do not consider 2-dimensional Calabi-Yau manifolds because some similarities to the 3-dimensional case get lost when going to even dimensions.

[2] The vanishing set of P_∞ has to be thought of as the vanishing set of $X(X-Z)Z$.

With the Frobenius method, we can easily find a basis of solutions for the Picard-Fuchs equation around each singular point. Around $z = 0$, we find the holomorphic and the singular solution

$$\begin{aligned} \varpi_0(z) &= \sum_{k=0}^{\infty} \binom{-1/2}{k}^2 z^k = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, z\right) \\ \varpi_1(z) &= \sum_{k=0}^{\infty} \binom{-1/2}{k}^2 \left(\log(z) + 4(H_{2k} - H_k)\right) z^k \end{aligned}$$

expressed with the harmonic numbers H_k. Both solutions have the convergence radius 1, which is the distance to the second singularity. Around $z = 1$, we find that we can write a basis of solutions as

$$\{\varpi_0(1-z), \varpi_1(1-z)\}$$

and around $z = \infty$ as

$$\left\{\frac{\varpi_0(1/z)}{\sqrt{z}}, \frac{\varpi_1(1/z)}{\sqrt{z}}\right\}.$$

To find the integer symplectic basis and the monodromy, we need to know how these bases are related after analytic continuation. For the Legendre curve, it is possible to calculate the exact transition matrices. Analytically continuing[3] on the upper half-plane we find that

$$\begin{aligned} \begin{pmatrix} \varpi_0(z) \\ \varpi_1(z) \end{pmatrix} &= \begin{pmatrix} \log(16)/\pi & -1/\pi \\ \log(16)^2/\pi - \pi & -\log(16)/\pi \end{pmatrix} \begin{pmatrix} \varpi_0(1-z) \\ \varpi_1(1-z) \end{pmatrix} \\ &= \begin{pmatrix} 1 + i\log(16)/\pi & -i/\pi \\ i\log(16)^2/\pi & 1 - i\log(16)/\pi \end{pmatrix} \frac{1}{\sqrt{z}} \begin{pmatrix} \varpi_0(1/z) \\ \varpi_1(1/z) \end{pmatrix}. \end{aligned}$$

We now want to relate the solutions of the Picard-Fuchs equation to actual periods given by integrating the holomorphic form over cycles from a basis $\{\gamma_1, \gamma_2\}$ of $H_1(X_z, \mathbb{Z})$. We know that any elliptic curve is a torus, and geometrically it is easy to fix two cycles that form a homology basis for the torus. To be able to exploit this, we visualise the elliptic curve as a double branched covering of $\mathbb{C}$ with branch points at $\{0, 1, z\}$ and the branch cuts chosen as in Figure 5.1.

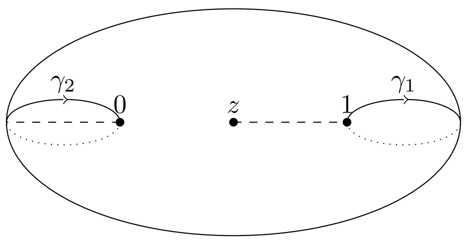

Figure 5.1: One sheet of the double branched covering of the Legendre curve.

We can see that γ_1 is invariant under transporting z around a closed path around $z = 0$ and that γ_2 is invariant under transporting z around a closed path around $z = 1$. Up to a rescaling, we hence find that the components of the period vector Π calculated with respect to γ_1 and γ_2 are uniquely given by the analytic continuations of

$$\begin{aligned} \Pi_1(z) &\sim \varpi_0(z) \\ \Pi_2(z) &\sim \varpi_0(1-z). \end{aligned}$$

[3]We always choose the branch of the logarithm such that the imaginary part is in the interval $(-\pi, \pi]$.

To find the relative constant between these solutions, we look at the monodromy. Going counter-clockwise around the singular point $z = 0$, we get the monodromy

$$\begin{pmatrix} \varpi_0(z) \\ \varpi_0(1-z) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -2i & 1 \end{pmatrix} \begin{pmatrix} \varpi_0(z) \\ \varpi_0(1-z) \end{pmatrix}$$

and going counter-clockwise around the singular point $z = 1$, we get

$$\begin{pmatrix} \varpi_0(z) \\ \varpi_0(1-z) \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi_0(z) \\ \varpi_0(1-z) \end{pmatrix}.$$

Up to an overall scaling, we thus find that an integer basis of periods is given by

$$\Pi(z) = \begin{pmatrix} \pi & 0 \\ i\log(16) & -i \end{pmatrix} \begin{pmatrix} \varpi_0(z) \\ \varpi_1(z) \end{pmatrix}.$$

In fact, it is not difficult to obtain the holomorphic solution around $z = 0$ by direct integration (see A.7) which shows that the scaling above is correct.

In the integer basis, the monodromies M_z for loops around singular points z have the form

$$M_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}.$$

As expected from the fact that composing the loops around 0 and 1 gives a loop which is homotopic to the loop around ∞, we have

$$M_0 M_1 = M_\infty.$$

We can also see that the complete monodromy group is the congruence subgroup $\Gamma(2)$, and we now want to explain the origin of it from a different perspective. The complex structure parameter z determines the complex structure

$$\tau(z) = \frac{\Pi_2(z)}{\Pi_1(z)} \in \mathbb{H}$$

of the torus, and the monodromy of the periods corresponds to $\Gamma(2)$ transformations of τ. Locally we can invert τ to get $z(\tau)$ which then gives a modular function under $\Gamma(2)$, also called a Hauptmodul in this context. For $|z(\tau)| < 1$, one can check that the holomorphic period around $z = 0$ gives rise to a Jacobi theta function

$$\vartheta_3(\tau)^2 = \frac{\Pi_1(z(\tau))}{\pi}$$

which is a modular form of weight 1 under $\Gamma(2)$. Here, the Picard-Fuchs equation is a differential equation of a modular form with respect to a Hauptmodul, and the monodromy group reflects the congruence subgroup with respect to which these are modular. Also note that the six elements in the quotient $SL(2,\mathbb{Z})/\Gamma(2)$ correspond to the fact that, given an elliptic curve, the associated z is not unique. This can be seen by calculating the j-invariant

$$j(z) = 256\frac{(1 + z(z-1))^3}{z^2(z-1)^2}.$$

For a more detailed discussion about the relation between differential equations and modular forms, we refer to Bruinier et al. (2008).

5.2 Example for Modularity

As an example, we take the member

$$X_2: \; ZY^2 = X(X-Z)(X-2Z)$$

of the Legendre curve and discuss the modularity, as explained in Section 4.3.

Modular Form and Periods

One can see that the reduction of X_2 to $\mathbb{F}_p$ is smooth for all primes $p \neq 2$. We know that the local zeta function has the form

$$\zeta(X_2/\mathbb{F}_p, T) = \frac{1 - a_p T + pT^2}{(1-pT)(1-T)}$$

and due to the modularity of elliptic curves, the a_p are coefficients of a Hecke eigenform $f \in S_2(\Gamma_0(N))$ with N being the conductor of the elliptic curve. By counting the number of points directly for some $\mathbb{F}_{p^k}$ or using the method of Dwork explained in Section 4.2, we can easily get the local zeta function for some primes p. Searching for the associated modular form with PARI/GP, one finds

$$f = (\eta(4\tau)\eta(8\tau))^2 \in S_2(\Gamma_0(32)).$$

In Section A.9, we give the associated meromorphic form $[F] \in \mathbb{S}_2(\Gamma_0(32))$ as well as the period polynomials. From the period polynomials, we can read off the periods $\omega^\pm$ and the quasiperiods $\eta^\pm$. By the modularity of elliptic curves, the values of the periods and its derivatives at $z=2$ are related to these periods and quasiperiods of f. Around $z=2$, the Frobenius basis for the Picard-Fuchs equation takes the form

$$\varpi(z) = \begin{pmatrix} 1 + \mathcal{O}(\delta^2) \\ \delta + \mathcal{O}(\delta^2) \end{pmatrix}$$

with $\delta = z - 2$. The Yukawa coupling is given by

$$C_z(z) = \frac{i\pi}{z(z-1)}.$$

This allows us to compute the exact intersection matrix in the local Frobenius basis

$$\Sigma_2 = \frac{i\pi}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Numerically, we calculate the transition matrix T which relates this basis to the integer basis by

$$\Pi = T\varpi.$$

We indeed find that we can express the entries by

$$T = \begin{pmatrix} -2\omega^+ + 2\omega^- & -\frac{1}{4}\eta^+ + \frac{1}{4}\eta^- \\ 2\omega^- & \frac{1}{4}\eta^- \end{pmatrix},$$

and the so-called Legendre relations

$$T^T \Sigma T = \Sigma_2$$

for T are equivalent to the Legendre relations for the periods and quasiperiods. The precision of the transition matrix has been tested with the Legendre relations which hold for at least 2000 decimal digits. The identification of the periods and quasiperiods holds for at least 2000 decimal digits, too. Our result also explains the choice of the meromorphic form we made in Section A.9, i.e. we have chosen it such that the first derivative of the periods Π only contains the quasiperiods. As an aside, we would like to note that the choice $z = 2$ is special since the elliptic curve then has complex multiplication. For the periods, this implies that there are relations between the real and imaginary part. In our case, we have

$$\begin{aligned}\omega^- &= -i\omega^+ \\ \eta^- &= i(\eta^+ - 2\omega^+).\end{aligned}$$

Additionally, invoking the Legendre relations, we can express the complete transition matrix in terms of the number $-2\omega^+$, which is known as the Lemniscate constant.

Modular Parametrisation

We follow Zagier (1985) to construct the modular parametrisation

$$\phi : X_0(32) \to X_2$$

explicitly. This should map ∞ to $(0:1:0)$, and the pullback $\phi^*\Omega$ should give a rational multiple of $2\pi i f \mathrm{d}\tau$. For our construction, we go to the chart $Z \neq 0$ and use inhomogeneous coordinates $(x, y) := \left(\frac{X}{Z}, \frac{Y}{Z}\right)$. Following Section A.7, we can express the holomorphic form by

$$\Omega = \frac{\mathrm{d}x}{2y}$$

for $y \neq 0$. The modular parametrisation can now be described by two holomorphic functions $g, h : \mathbb{H} \to \mathbb{C}$ modular under $\Gamma_0(32)$ such that

$$\phi(\tau) = (g(\tau) + 1 : h(\tau) : 1).$$

These functions are completely determined by the defining equation of the curve which gives

$$h^2 = g^3 - g \tag{5.1}$$

and the pullback of the holomorphic form

$$\frac{Dg}{2h} = f \in S_2(\Gamma_0(32)), \tag{5.2}$$

where the rational constant depends on the choice of the sign of h. These equations uniquely determine the coefficients in the q-expansion of g and h, which can be calculated easily. The first coefficients give

$$\begin{aligned}g &= \frac{1}{q^2} + q^2 + q^6 + \mathcal{O}(q^{18}) \\ h &= -\frac{1}{q^3} - q - 2q^5 - q^9 + \mathcal{O}(q^{17}).\end{aligned}$$

To prove that g and h are modular functions, one may identify these as rational functions in terms of modular forms. (5.1) and (5.2) can then be proved since these translate to relations in the finite-dimensional spaces of holomorphic modular forms.

Another possibility to get an analytic description of the modular parametrisations is to use the periods directly. To do this, one realises X_2 as $\mathbb{C}/\Lambda$ where Λ is a 2-dimensional lattice, whose complex structure is that of X_2. Choosing a basepoint $\tau_0 \in \overline{\mathbb{H}}$, the parametrisation can be defined by

$$\begin{aligned}\phi : X_0(N) &\to \mathbb{C}/\Lambda \\ \tau &\mapsto 2\pi i \int_\tau^{\tau_0} f(z)\mathrm{d}z.\end{aligned}$$

6 The Calabi-Yau Manifolds associated with AESZ34

We now turn to two one-parameter Calabi-Yau threefolds associated with the Calabi-Yau operator AESZ34 in the list by Almkvist et al. (2005). For the construction of these manifolds, we follow Hulek and Verrill (2005) and start with the vanishing set of

$$1 - z(X_0 + X_1 + X_2 + X_3 + X_4)\left(\frac{1}{X_0} + \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4}\right)$$

in $\mathbb{CP}^4 \backslash \{X_0X_1X_2X_3X_4 = 0\}$. Hulek and Verrill show that there is a resolution X_z which then is a Calabi-Yau manifold with Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 45 & & 0 & \\ 1 & & 5 & & 5 & & 1 \\ & 0 & & 45 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\ .$$

Following Candelas et al. (2019), the singular model features a symmetry group $G_1 \cong \mathbb{Z}/10\mathbb{Z}$ generated by the action

$$X_i \mapsto \frac{1}{X_{i+1}}$$

where the indices are to be understood mod 5, and this symmetry is also present for the resolution. This group further contains the subgroup $G_2 \cong \mathbb{Z}/5\mathbb{Z}$ generated by

$$X_i \mapsto X_{i+2}.$$

The quotients $X_z^\alpha := X_z/G_\alpha$ with $\alpha = 1, 2$ are Calabi-Yau manifolds with Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 4\alpha+1 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 4\alpha+1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

and hence the complex structure moduli space for these is 1-dimensional. Note that the group action of G_α is not fixed point free for all X_z and one finds that the quotients become singular for

$$z \in \Sigma := \left\{0, \frac{1}{25}, \frac{1}{9}, 1, \infty\right\}.$$

We derive the Picard-Fuchs operator $\mathcal{L}$ for X_z and its quotients in Section A.8. In the following, we will treat both manifolds simultaneously by leaving α as a parameter.

6.1 Special Points in the Moduli Space

We consider special points in the moduli space where X_z^α is modular, and the Hasse-Weil zeta function gives rise to modular forms.

Conifolds

A derivation of the Picard-Fuchs operator $\mathcal{L}$ can be found in A.8. It has the Riemann Symbol

$$\mathcal{P}\left\{\begin{array}{ccccc} 0 & \frac{1}{25} & \frac{1}{9} & 1 & \infty \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 2 \end{array}\right\}.$$

We can see that the singular points $\left\{\frac{1}{25}, \frac{1}{9}, 1,\right\}$ are conifolds and should correspond to rigid Calabi-Yau threefolds. Using the periods around the MUM-point $z = 0$, we compute the local zeta function as explained in Section 4.2. As expected, we find the factorisation

$$P_3(X_z^\alpha/\mathbb{F}_p, T) = (1 + \chi(p)pT)(1 - a_pT + p^3T^2).$$

For $z = 1/25$, this gives rise to the Hecke eigenform

$$q - 2q^2 + \mathcal{O}(q^3) \in S_4^{\text{new}}(\Gamma_0(30))$$

and for $z = 1/9$ and $z = 1$, one finds

$$\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2 \in S_4^{\text{new}}(\Gamma_0(6)).$$

Computing the periods and quasiperiods of these modular forms, one will find that they are rationally related to certain values in the transition matrix of the periods. This is explained in more detail for hypergeometric Picard-Fuchs operators by Klemm, Scheidegger, and Zagier (2020).

K-Point

The point $z = \infty$ is a K-point. Following Thorne (2018), K-points are typically associated with modular forms of weight 3. The deformation method does not work at $z = \infty$ and thus, if there is an associated modular form, we need a different strategy to find it. We discuss this in the next section.

Rank Two Attractor Points

We now discuss rank two attractor points of the two Calabi-Yau threefolds. Candelas et al. (2019) argue that the Hodge conjecture implies that these should occur at algebraic values of z, and they further conjecture that there are only finitely many. This means that we can describe the set of rank two attractor points by the roots of a polynomial

$$G(z) = c_0 + c_1z + ... + c_nz^n \tag{6.1}$$

with $c_i \in \mathbb{Z}$. The strategy Candelas et al. (2019) use to find this polynomial and the associated attractor points is to calculate the Frobenius $F_p(z)$ for all reduced parameters $z \in \mathbb{F}_p$ and many primes p and look for persistent factorisations of the form

$$P_3(X_z^\alpha/\mathbb{F}_p, T) = (1 - a_ppT + p^3T^2)(1 - b_pT + p^3T^2)$$

as explained in Section 4.3. If z_* is a rational rank two attractor point, one expects to see the factorisation for all primes p of good reduction and hence for all but finitely many primes. If z_* is irrational, we only get a contribution to the number of factorisations for primes p where z_* can be reduced to $\mathbb{F}_p$. For the latter case, one expects that the a_p and b_p give rise to more complicated automorphic forms.

In Figure 6.1, one can see how often $P_3(X_z^\alpha/\mathbb{F}_p, T)$ factorises for some primes p. One can see that it always factorises at least once, and this suggests the existence of a rational attractor point. Looking at the values, one indeed finds that it always factorises for

$$z = -\frac{1}{7}$$

and thus, a linear factor of G defined in (6.1) is $(7z+1)$. Motivated by this, Candelas et al. (2019) search for a quadratic factor and find factorisations for

$$z_\pm = 33 \pm 8\sqrt{17}.$$

Considering probabilities for random factorisation, they also argue why these may be the only rank two attractor points and thus

$$G(z) = (1+7z)(z^2 - 66z + 1)$$

would hold.

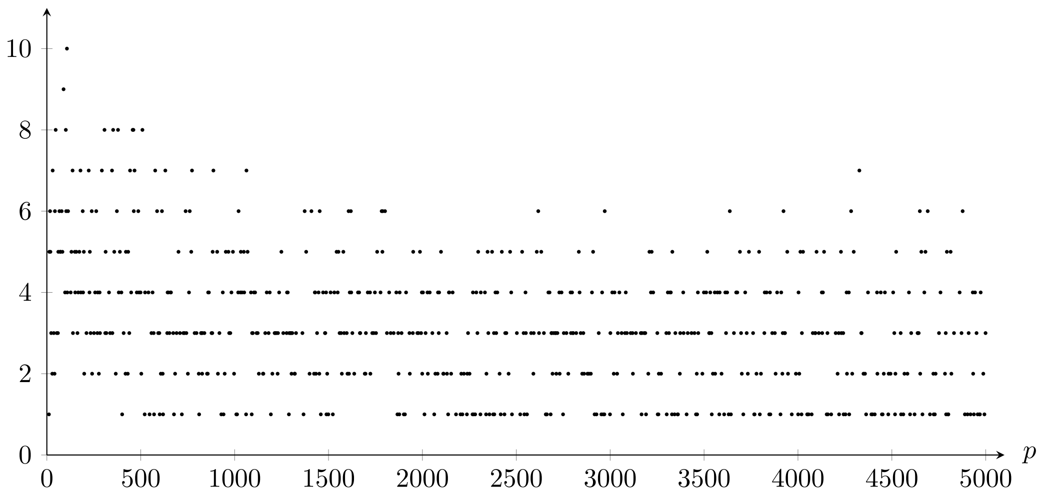

Figure 6.1: Number of factorisations of $P_3(X_z^\alpha/\mathbb{F}_p, T)$ into two quadratics for primes $7 < p < 5000$.

Searching for the Hecke eigenforms associated with the coefficients a_p and b_p at $z = -1/7$, one finds

$$\begin{aligned} f_2 &= \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau) \in S_2(\Gamma_0(14)) \\ f_4 &= q - 2q^2 + 8q^3 + 4q^4 + \mathcal{O}(q^5) \in S_4(\Gamma_0(14)). \end{aligned}$$

For the irrational rank two attractor points $z_\pm = 33 \pm 8\sqrt{17}$, one expects the occurrence of more complicated automorphic forms and, in this case, we have

$$\begin{aligned} f_2 &= q - q^2 + 2\sqrt{-2}q^3 + \mathcal{O}(q^4) \in S_2(\Gamma_0(34), \chi_{34}(33, \cdot)) \\ f_4 &= q - 2q^2 + 2\sqrt{-1}q^3 + \mathcal{O}(q^4) \in S_4(\Gamma_0(34), \chi_{34}(33, \cdot)), \end{aligned}$$

where the character is labelled by the Conrey label (see Bucur et al. (2018)).

6.2 Periods at the Rank Two Attractor Points and the K-Point

We label the Frobenius basis of periods around z_0 by ϖ_{z_0}. To go to an integer symplectic basis, we use the topological invariants of the mirror of X_z^α given by Candelas et al. (2019). With the scaling known from Section A.8, we define an integer symplectic basis in terms of the Frobenius basis around the MUM-point by

$$\Pi = (2\pi i)^3 \begin{pmatrix} \frac{-8\alpha\zeta(3)}{(2\pi i)^3} & \frac{\alpha}{4\pi i} & 0 & \frac{12\alpha}{(2\pi i)^3} \\ \frac{\alpha}{2} & 0 & -\frac{12\alpha}{(2\pi i)^2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 \end{pmatrix} \varpi_0.$$

We analytically continue the periods along the upper half-plane and define the transition matrices T_{z_0} for $\operatorname{Im} z_0 \geq 0$ by

$$\Pi = T_{z_0} \varpi_{z_0},$$

where the Frobenius basis around regular points z_0 is defined by

$$\varpi_{z_0} := \begin{pmatrix} 1 + \mathcal{O}(\delta^4) \\ \delta + \mathcal{O}(\delta^4) \\ \delta^2 + \mathcal{O}(\delta^4) \\ \delta^3 + \mathcal{O}(\delta^4) \end{pmatrix}$$

with $\delta = z - z_0$. We compute the Yukawa coupling as explained in Section 2.2 and find that

$$C_{zzz}(z) = (2\pi i)^3 \frac{12\alpha}{z^3(25z-1)(9z-1)(z-1)}.$$

This allows us to compute the exact intersection matrix Σ_{z_0} in the local Frobenius basis. The Legendre relations for the transition matrix then read

$$\Sigma_{z_0} = T_{z_0}^T \Sigma T_{z_0}.$$

In the following, we express the complete transition matrix for rank two attractor points in terms of the periods and quasiperiods of the associated modular forms. The meromorphic forms are chosen such that the periods of $\nabla_z^2\Omega$ and $\nabla_z^3\Omega$ do not contain the periods of the weight 2 form and the weight 4 form, respectively. For the K-point, we identify some entries in the transition matrix as periods and quasiperiods of a modular form and it is this comparison that allows us to identify the modular form. The precision of the transition matrix has been tested with the Legendre relations which hold for at least 2000 decimal digits. The identification of the periods and quasiperiods holds for at least 2000 decimal digits, too.

$z = -1/7$

For the rational attractor point, the intersection matrix in the local Frobenius basis is

$$\Sigma_{-1/7} = \alpha(2\pi i)^3 \frac{7^6}{2^{18}} \begin{pmatrix} 0 & 0 & 0 & 128 \\ 0 & 0 & -384 & -4004 \\ 0 & 384 & 0 & -12397 \\ -128 & 4004 & 12397 & 0 \end{pmatrix}.$$

The periods $\omega_k^\pm$ and quasiperiods $\eta_k^\pm$ for $k = 2, 4$ can be found in Section A.9. With these, we can write the transition matrix as

$$T_{-1/7} = \begin{pmatrix} -8\alpha & 0 & 343\alpha & 147\alpha \\ 30\alpha & 0 & -686\alpha & -294\alpha \\ 0 & 4 & 490 & 0 \\ -5 & 2 & 245 & 49 \end{pmatrix} \begin{pmatrix} \omega_4^+ & \eta_4^+ & 0 & 0 \\ \omega_4^- & \eta_4^- & 0 & 0 \\ 0 & 0 & \tilde{\omega}_2^+ & \tilde{\eta}_2^+ \\ 0 & 0 & \tilde{\omega}_2^- & \tilde{\eta}_2^- \end{pmatrix} \begin{pmatrix} -\frac{1}{6} & -\frac{35}{48} & -\frac{245}{64} & 0 \\ 0 & 0 & 0 & -\frac{1715}{98304} \\ 0 & \frac{1}{8} & 0 & -\frac{15337}{3072} \\ 0 & 0 & \frac{21}{1024} & \frac{7987}{32768} \end{pmatrix}$$

with $\tilde{\omega}_2^\pm := 2\pi i \omega_2^\pm$ and $\tilde{\eta}_2^\pm := 2\pi i \eta_2^\pm$. The Legendre relations for the transition matrix reduce to the Legendre relations of the periods and quasiperiods given in Section A.9.

$z_\pm = 33 \pm 8\sqrt{17}$

For the irrational attractor points, the intersection matrices in the local Frobenius basis are

$$\Sigma_{z_\pm} = \alpha(2\pi i)^3 \frac{1}{2^{21}}\{ \begin{pmatrix} 0 & 0 & 0 & -1140513311488 \\ 0 & 0 & 3421539934464 & -166820802991556 \\ 0 & -3421539934464 & 0 & -883858774734317 \\ 1140513311488 & 166820802991556 & 883858774734317 & 0 \end{pmatrix}$$
$$\pm\sqrt{17} \begin{pmatrix} 0 & 0 & 0 & -276615108864 \\ 0 & 0 & 829845326592 & -40459987722620 \\ 0 & -829845326592 & 0 & -214367240374035 \\ 276615108864 & 40459987722620 & 214367240374035 & 0 \end{pmatrix} \}.$$

The periods $\omega_k^\pm$ and quasiperiods $\eta_k^\pm$ for $k = 2, 4$ can be found in Section A.9. To write the transition matrix in terms of these, we now allow factors of $\sqrt{17}$ due to the irrationality of $z_\pm$ and factors of $\sqrt{-1}$ and $\sqrt{-2}$ because of the coefficient fields of the modular forms. We find that we can write the transition matrix as

$$T_{z_\pm} = A_\pm \begin{pmatrix} \tilde{\omega}_4^+ & \tilde{\eta}_4^+ & 0 & 0 \\ \tilde{\omega}_4^- & \tilde{\eta}_4^- & 0 & 0 \\ 0 & 0 & \tilde{\omega}_2^+ & \tilde{\eta}_2^+ \\ 0 & 0 & \tilde{\omega}_2^- & \tilde{\eta}_2^- \end{pmatrix} \begin{pmatrix} 1 & -\frac{85}{8} \pm \frac{175\sqrt{17}}{68} & \frac{62295}{136} \mp \frac{3555\sqrt{17}}{32} & 0 \\ 0 & 0 & 0 & -34270351 \pm 8311781\sqrt{17} \\ 0 & 1 & 0 & -\frac{614233}{26112} \pm \frac{2921\sqrt{17}}{512} \\ 0 & 0 & -3451 \pm 837\sqrt{17} & \frac{36995839}{192} \mp \frac{8972809\sqrt{17}}{192} \end{pmatrix}$$

with

$$A_+ = \begin{pmatrix} 4\alpha & 4\alpha & 9\alpha & -15\alpha \\ -30\alpha & -9\alpha & -16\alpha & 36\alpha \\ -30 & 7 & 20 & -15 \\ -5 & 4 & 9 & -11 \end{pmatrix}$$

and

$$A_- = \begin{pmatrix} 2\alpha & 0 & 0 & 6\alpha \\ 0 & -3\alpha & 2\alpha & 0 \\ 0 & -1 & -5 & 0 \\ -5 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -\sqrt{-1} & 0 & 0 & 0 \\ 0 & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{-2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{-2}} \end{pmatrix}.$$

Here, we also defined the rescaled periods and quasiperiods by

$$\begin{pmatrix} \tilde{\omega}_2^+ & \tilde{\eta}_2^+ \\ \tilde{\omega}_2^- & \tilde{\eta}_2^- \end{pmatrix} := 2\pi i \frac{9(35904 - 10081\sqrt{-2} \mp (8708 - 2445\sqrt{-2})\sqrt{17})}{1088} \begin{pmatrix} \tilde{\omega}_2^+ & \tilde{\eta}_2^+ \\ \tilde{\omega}_2^- & \tilde{\eta}_2^- \end{pmatrix} C_2$$
$$\begin{pmatrix} \tilde{\omega}_4^+ & \tilde{\eta}_4^+ \\ \tilde{\omega}_4^- & \tilde{\eta}_4^- \end{pmatrix} := \frac{21 + 103\sqrt{-1} \mp (5 + 25\sqrt{-1})\sqrt{17}}{2890} \begin{pmatrix} \tilde{\omega}_4^+ & \tilde{\eta}_4^+ \\ \tilde{\omega}_4^- & \tilde{\eta}_4^- \end{pmatrix} C_4$$

with

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{135}{7343982592} - \frac{5\sqrt{-1}}{7802981504} \end{pmatrix}$$
$$C_4 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-88731327636+65414683821\sqrt{-2}}{140579843017716458497834} \end{pmatrix}$$

where the sign choice of $\sqrt{17}$ depends, as in the definition of the periods and quasiperiods in Section A.9, on the attractor point $z_\pm$ we consider. The Legendre relations for the transition matrix again reduce to the Legendre relations of the periods and quasiperiods given in Section A.9.

$z = \infty$

For the K-point $z = \infty$, there may exist an associated modular form of weight 3[1]. To find this, we compute the transition matrix T_∞ relating the Frobenius basis

$$\varpi_\infty(\delta) := \begin{pmatrix} f_0(\delta) \\ f_0(\delta)\log(\delta) + f_1(\delta) \\ f_2(\delta) \\ f_2(\delta)\log(\delta) + f_3(\delta) \end{pmatrix}$$

with $f_0(\delta) = \delta + \mathcal{O}(\delta^2)$, $f_1(\delta) = \mathcal{O}(\delta^2)$, $f_2(\delta) = \delta^2 + \mathcal{O}(\delta^3)$ and $f_3(\delta) = \mathcal{O}(\delta^3)$ to the integer basis Π. We then compute the periods of modular forms in $S_3^{\mathrm{new}}(\Gamma_0(N), \chi)$ for low levels N and characters χ and search for equality to values in the transition matrix up to rational factors. This search suggests that the associated modular form is given by

$$f = q + q^2 + \mathcal{O}(q^3) \in S_3^{\mathrm{new}}(\Gamma_0(15), \chi_{15}(14, \cdot)).$$

We can express the second and the fourth column of the transition matrix by

$$T_{\cdot,2} = \begin{pmatrix} \alpha(\omega^- + 2\omega^+) \\ \alpha(-2\omega^- - 6\omega^+) \\ 2\omega^- \\ \omega^- + \omega^+ \end{pmatrix}, \quad T_{\cdot,4} = \begin{pmatrix} \alpha(\eta^- + 2\eta^+) \\ \alpha(-2\eta^- - 6\eta^+) \\ 2\eta^- \\ \eta^- + \eta^+ \end{pmatrix}.$$

Here, $\omega^+ \in \mathbb{R}$ and $\omega^- \in i\mathbb{R}$ are the periods of f with the normalisation defined by $r_f = \omega^+ r^+ + \omega^- r^-$ with

$$r^+\left(\begin{pmatrix} 16 & -3 \\ 75 & -14 \end{pmatrix}\right) = \frac{15}{2}\tau - \frac{3}{2}, \quad r^-\left(\begin{pmatrix} 16 & -3 \\ 75 & -14 \end{pmatrix}\right) = -5\tau + 1.$$

We did not explicitly construct a meromorphic form associated with f, but the $\eta^\pm$ should be the quasiperiods of f. With the intersection matrix

$$\Sigma_\infty = \alpha(2\pi i)^3 \frac{2^2}{3^3 5^4} \begin{pmatrix} 0 & 0 & 450 & -225 \\ 0 & 0 & 225 & 0 \\ -450 & -225 & 0 & -26 \\ 225 & 0 & 26 & 0 \end{pmatrix},$$

we find that

$$\frac{1}{(2\pi i)^2}(\omega^+\eta^- - \omega^-\eta^+) = \frac{4}{5\sqrt{-15}}.$$

[1]The character must be non-trivial, since there are no holomorphic modular forms with odd weight and trivial character.

7 Rank Two Attractor Points of Self Hadamard Products

In this chapter, we consider special Calabi-Yau operators constructed as the self Hadamard products of degree two Calabi-Yau operators. The operators feature an involutional symmetry where one of the fixed points is both an apparent singularity and a rank two attractor point. The deformation method for the computation of the zeta function does not work at apparent singularities and we use the numerical values of the periods to compute the zeta function by finding the associated modular forms.

7.1 The Hadamard Product

For the definition of the Hadamard product of families of varieties, we follow Samol (2010). Let $X \to \mathbb{CP}^1$ and $Y \to \mathbb{CP}^1$ be two families of varieties. Further, let Z be the blowup of $\mathbb{CP}^1 \times \mathbb{CP}^1$ in $(0, \infty)$ and $(\infty, 0)$. Then there exists a compactification

$$\mu : Z \to \mathbb{CP}^1$$

of the multiplication map

$$\begin{aligned} \mathbb{C}^* \times \mathbb{C}^* &\to \mathbb{C}^* \\ (s, t) &\mapsto s \cdot t. \end{aligned}$$

Using this, we define a new family $X \star Y \to \mathbb{CP}^1$ by pulling back $X \times Y \to \mathbb{CP}^1 \times \mathbb{CP}^1$ with the projection $Z \to \mathbb{CP}^1 \times \mathbb{CP}^1$:

$$\begin{array}{ccc} X \star Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathbb{CP}^1 \times \mathbb{CP}^1 \\ \mu \downarrow & & \\ \mathbb{CP}^1 & & \end{array}$$

We call $X \star Y$ the Hadamard product of X and Y and one can easily see that the fibre has dimension $\dim X + \dim Y + 1$. We now want to see what the Hadamard product does on the level of the periods. To simplify things, we consider the case that locally X_s and Y_t are defined by the hypersurfaces

$$\begin{aligned} F_s(x) &= 0 \\ G_t(y) &= 0. \end{aligned}$$

The fibre $(X \star Y)_z$ is then given by all points (x, y, s, t) satisfying

$$F_s(x) = 0 \ , \quad G_t(y) = 0 \ , \quad st - z = 0.$$

In the spirit of Section A.6, we further assume that there are holomorphic periods

$$\begin{aligned} f(s) &= \int_{T(\gamma_X)} \frac{\eta_X}{F_s} = \sum_{k=0}^{\infty} f_k s^k \\ g(t) &= \int_{T(\gamma_Y)} \frac{\eta_Y}{G_t} = \sum_{k=0}^{\infty} g_k t^k. \end{aligned}$$

Letting $\{(s,t) \in S^1 \times S^1\}$ denote a tube around $st = z$, we can write down a period of the Hadamard product as

$$\begin{aligned}\Pi(z) &= \frac{1}{(2\pi i)^2}\int_{T(\gamma_X)\times T(\gamma_Y)\times S^1\times S^1} \frac{\eta_X\wedge\eta_Y\wedge \mathrm{d}s\wedge \mathrm{d}t}{F_s G_t(st-z)} \\ &= \frac{1}{(2\pi i)^2}\sum_{k,l=0}^{\infty} f_k g_l \int_{S^1\times S^1}\frac{s^k t^l}{st-z}\mathrm{d}s\wedge\mathrm{d}t \\ &= \sum_{k=0}^{\infty} f_k g_k z^k =: (f\star g)(z).\end{aligned}$$

If f and g are annihilated by differential operators $\mathcal{L}_X$ and $\mathcal{L}_Y$ of Fuchsian type, then the Hadamard product $f\star g$ is also annihilated by a differential operator of Fuchsian type. A proof for this can be found in Stanley (1999, p. 194). We define $\mathcal{L}_X \star \mathcal{L}_Y$ to be the lowest order Fuchsian operator annihilating $f\star g$. In general, the Hadamard product of two Calabi-Yau operators does not have to be a Calabi-Yau operator again. However, there are examples of fourth order Calabi-Yau operators which are the Hadamard product of second order Calabi-Yau operators. In the following, we restrict to fourth order Calabi-Yau operators which are self Hadamard products, i.e.

$$\mathcal{L} = \mathcal{L}_X \star \mathcal{L}_X$$

for some second order Calabi-Yau operator $\mathcal{L}_X$.

7.2 Rank Two Attractor Points

Let $E \to \mathbb{CP}^1\backslash\Sigma$ be a family of elliptic curves with a symmetry

$$E_s \cong E_{\frac{1}{\gamma s}} \tag{7.1}$$

for some $\gamma\in\mathbb{Z}$. Now consider the self Hadamard product $X = E\star E$ with the fibre X_z given by

$$(x,y)\in E_s\times E_t\ ,\quad s\cdot t = z.$$

There is an obvious isomorphism

$$X_z \cong X_{\frac{1}{\gamma^2 z}}$$

induced by the involution (7.1). This also exists for other Hadamard products, but for simplicity we restrict to self Hadamard products. The isomorphism acts on the cohomology, and for the fixed points $z = \pm\frac{1}{\gamma}$ this may split the middle cohomology into two parts with fixed Hodge structure. A splitting at this point was first observed by Elmi (2020) and, in the following, we consider ten examples where the splitting happens over $\mathbb{Q}$ and the varieties are rank two attractor varieties. In these examples, one fixed point is a conifold singularity while the other one, the rank two attractor point z_*, is an apparent singularity of the Picard-Fuchs equation with local exponents $\{0,1,3,4\}$. This point is rational and hence we expect to find modular forms of weight 4 and 2 under $\Gamma_0(N)$ for some level N. However, at apparent singularities, the deformation method explained in Section 4.2 does not work. We do the following to find the modular forms:

- We use (2.2) to numerically find a basis in which the monodromy is rational.
- We read of the transcendental numbers $\omega_4^+\in\mathbb{R}$ and $\omega_4^-\in i\mathbb{R}$ appearing in $\Pi(z_*)$. The same is done for the projection $\Pi^{2,1}\Pi'(z_*)$ to read of $\omega_2^+\in\mathbb{R}$ and $\omega_2^-\in i\mathbb{R}$.
- Starting with low level N, we compute the periods of all Hecke eigenforms in $S_2^{\mathrm{new}}(\Gamma_0(N))$ and $S_4^{\mathrm{new}}(\Gamma_0(N))$ until we find rational equivalence to $\omega_4^\pm$ and $\omega_2^\pm$. These forms are then our final candidates.

The modular forms we found in this way are listed in Table 7.1. For the identification, we worked with a precision of at least 2000 decimal digits. The Calabi-Yau operator associated with the AESZ label can be found in the list of Almkvist et al. (2005).

AESZ	z_*	N_2	N_4	Constraints on f_2	Constraints on f_4
100	1/8	14	14		$a_3 = -2$
101	1	11	22		$a_3 = -7$
103	-1/9	90	180	$(a_7, a_{11}) = (-4, 0)$	$(a_7, a_{11}) = (-28, 24)$
107	-1/32	48	48		$a_5 = 6$
115	-1/256	32	32		$a_3 = 8$
144	-1/72	306	306	$(a_5, a_7, a_{23}) = (0, 2, 6)$	$a_5 = 12$
145	-1/729	54	108	$a_5 = -3$	$(a_5, a_7) = (9, -1)$
155	-1/4096	128	128	$(a_3, a_5) = (-2, 2)$	$(a_3, a_5) = (-2, -6)$
165	-1/27	27	54		$a_5 = 12$
166	-1/186624	864	864	$(a_5, a_7, a_{11}) = (1, -3, -3)$	$(a_5, a_7, a_{11}) = (-19, 13, 65)$

Table 7.1: Sufficiently many constraints on the Hecke eigenvalues such that $f_2 \in S_2^{\text{new}}(\Gamma_0(N_2))$ and $f_4 \in S_4^{\text{new}}(\Gamma_0(N_4))$ appearing at the rank two attractor z_* are uniquely determined.

The unit root method of Dwork, introduced for the Legendre curve in Section 4.2, is applicable to one-parameter Calabi-Yau threefolds - Samol (2010) gives an application of this. Candelas (2020) offered us an implementation of this method. We observe that the method works at apparent singularities and gives rise to the coefficient of the associated weight 4 forms. Exemplary, we checked the coefficients for all primes of good reduction out of $\{5, 7, 11, 13\}$. These always coincide with the coefficients of the modular forms we found.

It should be noted that the calculation of the topological invariants of the varieties by demanding integral monodromy yields positive Euler numbers of the mirror for the last seven examples (107 - 166), which suggests that these cannot be one-parameter models.

8 Conclusion

After an introduction to the geometry of Calabi-Yau manifolds and the theory of modular forms and period polynomials, we have explained how algebraic varieties can be reduced to finite fields and how this gives rise to the Hasse-Weil zeta function. We have shown how this can relate Calabi-Yau manifolds and modular forms for the examples of elliptic curves, rigid Calabi-Yau threefolds, and rank two attractor varieties. We then started with the example of the Legendre family of elliptic curves. After a general discussion of the global structure of the periods, we went to one specific member of the family and found the associated modular form. As expected, the periods of the holomorphic form and their derivatives were rationally related to the periods and quasiperiods of the modular form, because there exists a modular parametrisation of the elliptic curve. We have shown two ways to explicitly obtain such a modular parametrisation: by constructing meromorphic functions and by giving an analytic description in terms of integrals of the modular form.

We reviewed how one can find three attractor points in the complex structure moduli space of two Calabi-Yau threefolds associated with the Calabi-Yau operator AESZ34. One of the attractor points is rational and gives rise to holomorphic modular forms under $\Gamma_0(14)$ with trivial character. We calculate the periods and quasiperiods of these forms and show that the periods of the holomorphic form and all of their derivatives are rationally related to these. The two irrational attractor points give rise to modular forms under $\Gamma_0(34)$ with non-trivial character. For the computation of the periods and quasiperiods, we treat these as modular forms with trivial character under $\Gamma_1(34)$. We again find that the periods of the holomorphic form and all of their derivatives can be written as linear combinations of the periods and quasiperiods of the modular forms. In this case, this does not work over $\mathbb{Q}$, but we have to extend by $\sqrt{17}$, $\sqrt{-2}$ and $\sqrt{-1}$ due to the irrationality of the attractor points and the coefficient fields of the modular forms. For all three attractor points, we calculate the intersection matrix in the local Frobenius basis and find that the Legendre relations of the transition matrix are equivalent to the quadratic Legendre relations between the periods and quasiperiods. At the K-point $z = \infty$, the deformation method does not work and we identified a modular form of weight 3 associated with this point by comparing the numerical values of the periods and their derivatives to periods of modular forms.

We considered ten degree four Calabi-Yau operators corresponding to self Hadamard products of degree two Calabi-Yau operators. These feature an involutional symmetry where one of the fixed points is an apparent singularity. This fixed point is also a rank two attractor point, but due to the presence of the apparent singularity, the deformation method cannot be used to calculate local zeta functions. We determine the associated modular forms and thus also the local zeta functions by comparing the numerical values of the periods of the holomorphic form to the periods and quasiperiods of modular forms of low level. For all of the considered cases, this gives unique candidates for the modular forms and we have good evidence suggesting that these are correct.

The topic of rank two attractor points leaves much space for further research in physics and mathematics. For example, they are also used for flux compactifications in string theory (see Kachru, Nally, and Yang (2020)), and we believe that they are associated with conformal supergravities. Another related field is that of Feynman integrals in quantum field theory. Following Klemm, Nega, and Safari (2019), these can sometimes be realised as integrals over the holomorphic form of Calabi-Yau manifolds, e.g. there is a Feynman diagram associated with AESZ34. For special configurations of the parameters of the Feynman diagram, the underlying Calabi-Yau manifold is a rank two attractor variety and thus the periods and quasiperiods of modular forms appear in physical amplitudes. Mathematically, one ambiguous goal would be to find correspondences between rank two attractor varieties and Kuga-Sato threefolds, which would ultimately explain the numerical observations we made. Other possible research directions are given by generalisations to multi-parameter models and more complicated forms of modularity of Calabi-Yau threefolds.

A Appendix

A.1 The Frobenius Method

The Frobenius method allows us to find expansions for the solutions of differential equations of Fuchsian type around every point $z \in \mathbb{CP}^1$. We give a short overview on how to do this and refer to Ince (2008) for more details. We consider a Fuchsian differential operator

$$\mathcal{L} = \sum_{k=0}^{n} P_k(z)\Theta^k$$

with $\Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$. Suppose we want to find the solutions of the differential equation

$$\mathcal{L}f = 0$$

around some point, say $z = 0$. For this, we first look at solutions of the indicial equation

$$\sum_{k=0}^{n} P_k(0)r^k = 0.$$

The solutions $\{r_1, ..., r_n\}$ of this equation are called the local exponents, and for each local exponent r one can find a holomorphic solution of the form

$$f(z) = z^r \sum_{k=0}^{\infty} a_k z^k$$

by solving the recursion relation for the (not necessarily unique) coefficients a_k. At singular points z_i, it can happen that the indicial equation has multiple roots. To cover this case, assume that at some point, say $z = 0$, we have a local exponent r with multiplicity n_r. Then there are power series $g_0, ..., g_{n_r-1}$ with leading exponents $\geq r$ that give rise to n_r solutions of the form

$$f_i(z) = \sum_{k=0}^{i} \frac{(\log z)^k}{k!} g_{i-k}(z)$$

for $i = 0, ..., n_r - 1$. In particular, the local exponents give the behaviour of the solutions for $z \to 0$ and the form of the monodromy around $z = 0$. Hence, it is useful to store this information for all singular values $z_1, ..., z_m$ of the differential equation in the Riemann symbol

$$\mathcal{P}\left\{\begin{array}{ccc} z_1 & \cdots & z_m \\ \hline r_{1_1} & \cdots & r_{m_1} \\ \vdots & \ddots & \vdots \\ r_{1_n} & \cdots & r_{m_n} \end{array}\right\}.$$

Note that there can be singular points z_i of the differential equations at which none of the solutions is singular. We call these points apparent singularities.

A.2 Representatives for $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$

We want to find representatives for the quotient $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ with

$$\mathcal{M}_{n,N} = \{\gamma \in M_2(\mathbb{Z}) |\ \det\gamma = n,\ c_\gamma \equiv 0 \bmod N\}$$

and $(n, N) = 1$. We first show that we can choose the representatives to satisfy $c_\gamma = 0$. For this, we start with a general element

$$\begin{pmatrix} a & b \\ kN & d \end{pmatrix} \in \mathcal{M}_{n,N}$$

and multiply from the left with

$$\begin{pmatrix} x & y \\ -\frac{k}{(a,k)}N & \frac{a}{(a,k)} \end{pmatrix}.$$

The resulting element has $c_\gamma = 0$, but we still need to show that there are $x, y \in \mathbb{Z}$ such that

$$\begin{pmatrix} x & y \\ -\frac{k}{(a,k)}N & \frac{a}{(a,k)} \end{pmatrix} \in \Gamma_0(N),$$

i.e.

$$x\frac{a}{(a,k)} + y\frac{k}{(a,k)}N = 1.$$

This is a linear Diophantine equation which has solutions if and only if $(a, kN) = (a, k)$. This is true since

$$ad - bkN = n$$

implies that $(a, N)|n$ and since $(n, N) = 1$ we have $(a, N) = 1$. Thus, we have shown that any element in the quotient $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ can be represented by elements of the form

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Since $-1 \in \Gamma_0(N)$, we can demand that $d > 0$. Using the translation operator T with

$$T^k\gamma = \begin{pmatrix} a & b + kd \\ 0 & d \end{pmatrix},$$

we further find that we can restrict to $0 \leq b < d$. We conclude that a generating set of $\Gamma_0(N)\backslash\mathcal{M}_{n,N}$ is given by

$$\{\gamma \in M_2(\mathbb{Z}) |\ a_\gamma d_\gamma = n,\ 0 \leq b_\gamma < d_\gamma\}.$$

Checking that this generating set is minimal is trivial. This generating set does not depend on the level N, and one can now easily see that the cardinality of the quotient is given by

$$|\Gamma_0(N)\backslash\mathcal{M}_{n,N}| = \sigma_1(n),$$

i.e. it is given by the sum of divisors of n.

A.3 Conventions for Period Polynomials

For our application, we will need to calculate period polynomials for weight $k = 2$ and $k = 4$. For $k = 2$, there is no ambiguity and the period polynomial for f holomorphic or meromorphic is uniquely given by

$$r_f(\gamma) = 2\pi i \int_{\tau_0}^{\gamma^{-1}\tau_0} f(z)\mathrm{d}z$$

and does not depend on the choice of τ_0. For $k > 2$, we fix the representative of the period polynomial by choosing $\tau_0 = \infty$ in (3.4). Following Klemm, Scheidegger, and Zagier (2020), the corresponding Eichler integral $\tilde{f}$ can then also be expressed by

$$\tilde{f}(\tau) = \frac{(2\pi i)^{k-1}}{(k-2)!} \left(\int_{\tau_0}^{\tau} (z-\tau)^{k-2} f(z)\mathrm{d}z + \frac{1}{k-1} \int_{\tau_0}^{\tau_0+1} B_{k-1}(z-\tau) f(z)\mathrm{d}z \right)$$

and this expression does not depend on τ_0 because the Bernoulli polynomials satisfy

$$B_n(x+1) = B_n(x) + nx^{n-1}.$$

This is very convenient for practical calculations, because τ_0 can be chosen such that this integral converges quickly without the need to expand f to very high order. However, in general this does not work for meromorphic F since the constant term of the boundary polynomial will depend on the path of integration. In this case, we can use the rational polynomials $r^{\pm}$ obtained from the holomorphic modular form f. When γ is chosen such that there are no $p^{\pm} \in V_{k-2}^{\mathbb{C}}$ satisfying

$$p^{\pm}|_{2-k}(\gamma - 1) = r^{\pm}(\gamma),$$

we can choose the boundary term uniquely by requiring proportionality of the period polynomial of F to $r^{\pm}(\gamma)$.

A.4 Finding Meromorphic Forms

To find a meromorphic form $F \in \mathbb{S}_k(\Gamma_0(N), \chi)$ associated with a Hecke eigenform $f \in S_k(\Gamma_0(N), \chi)$, we make an ansatz for F and the boundary terms in $D^{k-1} M_{2-k}^{\mathrm{mero}}(\Gamma_0(N), \chi)$. First, we write

$$F = \frac{G}{h}$$

with $G \in S_{k+k_h}(\Gamma_0(N), \chi)$ and $h \in M_{k_h}(\Gamma_0(N))$. The hope then is that for k_h high enough, we can find the correct meromorphic form associated with f. To find a form with correct eigenvalues under T_n, we consider for the boundary terms elements

$$D^{k-1} \frac{g}{h^n}$$

with

$$g \in M_{2-k+n\cdot k_h}(\Gamma_0(N), \chi).$$

This can be motivated by looking at the pole order of the elements after applying T_n. Checking whether our ansatz gives a possible candidate for the meromorphic form reduces to linear algebra. The computational effort can be reduced by considering the correct Atkin-Lehner eigenvalues of G and h. To be able to check the vanishing residues of F at all poles, we will restrict h to have zeros at the cusps, i.e. we construct h as an eta-quotient. For this, we follow Rouse and Webb (2015).

A.5 Finite Fields and p-adic Numbers

Recall that a field is a set $\mathbb{K}$ with the operations $+$ and $\cdot$ with the following properties:

- $(\mathbb{K}, +)$ is an abelian group whose neutral element we call 0.
- $(\mathbb{K}^*, \cdot) = (\mathbb{K}\backslash\{0\}, \cdot)$ is an abelian group whose neutral element we call 1.
- The distributive laws[1]

$$\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c \\ (a + b) \cdot c &= a \cdot c + b \cdot c \end{aligned}$$

  hold for all $a, b, c \in \mathbb{K}$.

Common examples which are not finite are $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$. The characteristic of a field is defined as the smallest $n > 0$ such that

$$\underbrace{1 + ... + 1}_{n\text{-times}} = 0$$

if such an n exists. Otherwise, it is defined to be zero. All finite fields must have a characteristic $n \neq 0$ and one can easily show that the characteristic must be a prime number. Further, one can show that any finite field with characteristic p must have order p^k for some $k > 0$ and that these are unique up to isomorphisms. We denote a field with p^k elements by $\mathbb{F}_{p^k}$. For $k = 1$, one can choose

$$\mathbb{F}_p = \{0, ..., p - 1\}$$

with addition and multiplication modulo p. To have a representative for $k > 1$, let ρ denote a root of a monic irreducible polynomial of degree k in $\mathbb{F}_p$. We can then write

$$\mathbb{F}_{p^k} = \{a_0 + a_1\rho + ... + a_{k-1}\rho^{k-1} | a_i \in \mathbb{F}_p\}$$

with the obvious multiplication and addition.

Another field of interest for us are the p-adic numbers $\mathbb{Q}_p$ for primes p. These are the completion of the rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$. This norm is defined by

$$|x|_p := p^{-\alpha_p(x)}$$

where $\alpha_p(x)$ is the exponent of p in the prime factorisation of x. It can easily be shown that this defines a non-Archimedean, i.e. it is a norm which further satisfies

$$|x + y|_p \leq \max(|x|_p, |y|_p) \ \forall x, y \in \mathbb{Q}.$$

In pratice, it is convenient to think about a p-adic number $x \in \mathbb{Q}_p$ as a unique expansion

$$x = \sum_{k=m}^{\infty} a_n p^k$$

with $a_k \in \{0, 1, ..., p - 1\}$, $m \in \mathbb{Z}$, and $a_m \neq 0$. The p-adic norm is then given by $|x|_p = p^{-m}$. An important subset of the p-adic numbers are the p-adic integers defined by

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p | \ |x|_p \leq 1\}.$$

For $z \in \mathbb{F}_p$, the Teichmüller lift is the unique $\tilde{z} \in \mathbb{Z}_p$ such that

$$\tilde{z}^p = \tilde{z} \quad \text{and} \quad \tilde{z} \equiv z \bmod p.$$

[1]As usual, the operation $\cdot$ has higher precedence than $+$.

A.6 Representing Period Integrals

For n-dimensional Calabi-Yau manifolds X_z, which are described as the vanishing set of a polynomial P_z homogeneous of degree $n+2$ in $\mathbb{CP}^{n+1}$ and with complex structure parameter z, one can represent integrals of the holomorphic form over cycles $\gamma_z \in H_n(X_z, \mathbb{Z})$ by higher-dimensional integrals in the ambient space $\mathbb{CP}^{n+1}$. For this, we use the Leray coboundary map

$$T_z : H_n(X_z, \mathbb{Z}) \to H_{n+1}(\mathbb{CP}^{n+1} \backslash X_z, \mathbb{Z}),$$

which assigns a unique tube $T_z(\gamma_z)$ to the cycle γ_z. With projective coordinates $(X_0, ..., X_{n+1})$ of $\mathbb{CP}^{n+1}$, we symbolically define

$$\mu = \frac{1}{2\pi i} \sum_{i=0}^{n+1} (-1)^i X_i \mathrm{d}X_0 \wedge ... \wedge \widehat{\mathrm{d}X_i} \wedge ... \wedge \mathrm{d}X_{n+1},$$

which enables us to form well-defined integrals

$$\int_{T_z(\gamma_z)} \frac{f}{P_z^k} \mu \tag{A.1}$$

for polynomials f homogeneous of degree $(n+2)(k-1)$. Of course, not all choices of f and k give different integrals. If f can be decomposed in the Jacobian ideal J_{P_z}

$$f = \sum_{i=0}^{n+1} f_i \partial_i P_z,$$

we can use Stokes theorem to get

$$\sum_{i=0}^{n+1} \int_{T_z(\gamma_z)} \frac{f_i \partial_i P_z}{P_z^k} \mu = \frac{1}{k-1} \sum_{i=0}^{n+1} \int_{T_z(\gamma_z)} \frac{\partial_i f_i}{P_z^{k-1}} \mu.$$

This reduction of the pole order is also known as Griffiths–Dwork reduction.

For odd n, one can show that all integrals of forms in the middle cohomology, i.e. forms in $H^{n-k,k}(X_z)$, can be represented by the integral (A.1)[2]. Together with the Griffiths-Dwork reduction, this then establishes a group isomorphism

$$\mathbb{C}[X_0, ..., X_{n+1}]_{(n+2)(k-1)} / J_{P_z} \cong H^{n-k,k}(X_z).$$

Normalizing the holomorphic form accordingly, we can directly write down the period with respect to γ_i as

$$\Pi_i(z) = \int_{\gamma_i(z)} \Omega_z = \int_{T_z(\gamma_i(z))} \frac{1}{P_z} \mu. \tag{A.2}$$

As claimed before, we see that with this choice, the periods are holomorphic in z. Further, one can locally choose the Leray coboundary map T to be constant in the sense that for z' close to z the elements $T_z(\gamma_i(z))$ and $T_{z'}(\gamma_i(z'))$ are equal and lie in

$$H_{n+1}(\mathbb{CP}^{n+1} \backslash (X_z \cup X_{z'}), \mathbb{Z}).$$

This means that if we differentiate the expression (A.2), we only need to differentiate the integrand. This allows to obtain the Picard-Fuchs equation by decomposing polynomials in certain ideals and using the Griffiths-Dwork reduction.

[2] For general n, this only gives the primitive part of the cohomology.

A.7 Holomorphic Period for the Legendre Family

For the Legendre family, we can calculate the period associated with γ_1 in Figure 5.1 by direct integration. Note that this corresponds to the holomorphic solution of the Picard-Fuchs equation around $z = 0$, because it has trivial monodromy around that point. For simplicity, let $0 < z < 1$ and $z \in \mathbb{R}$. For the integration, we go to the patch $Z = 1$ such that

$$\Pi_1(z) = \frac{1}{2\pi i} \int_{T(\gamma_1)} \frac{1}{Y^2 - X(X-1)(X-z)} \mathrm{d}X \wedge \mathrm{d}Y.$$

The path γ_1 corresponds to X going from 1 to ∞ on the upper sheet and returning on the lower one. Thus, we see that the tube around that path can be realised by the points (X, Y) in

$$\{(X, \pm\sqrt{X(X-1)(X-z)} + \delta) \in \mathbb{C}^2 |\ X \in [0, \infty) \text{ and } |\delta| = \epsilon\}$$

with $0 < \epsilon \ll 1$. The contributions from both sheets are equal and hence

$$\begin{aligned}
\Pi_1(z) &= \frac{1}{\pi i} \int_1^\infty \mathrm{d}X \oint \mathrm{d}\delta \frac{1}{\delta^2 + 2\delta\sqrt{X(X-1)(X-z)}} \\
&= \int_1^\infty \mathrm{d}X \frac{1}{\sqrt{X(X-1)(X-z)}} \\
&= \sum_{k=0}^\infty \binom{-1/2}{k} (-z)^k \int_1^\infty \mathrm{d}X \frac{1}{X^{k+1}\sqrt{X-1}} \\
&= \pi \sum_{k=0}^\infty \binom{-1/2}{k}^2 z^k.
\end{aligned}$$

A.8 Obtaining the Picard-Fuchs Operator AESZ34

Away from the singularities, we can describe the manifold X_z underlying AESZ34 by a quotient of the vanishing set of

$$P_z(X) = X_0X_1X_2X_3X_4 \left(1 - z(X_0 + X_1 + X_2 + X_3 + X_4) \left(\frac{1}{X_0} + \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4}\right)\right)$$

in $\mathbb{CP}^4$. We go to the patch with $X_0 \neq 0$ constant and want to compute the period

$$\Pi_1(z) = \frac{1}{2\pi i} \int_{T(\gamma_1)} \frac{1}{P_z(X)} X_0 \mathrm{d}X_1 \wedge \mathrm{d}X_2 \wedge \mathrm{d}X_3 \wedge \mathrm{d}X_4$$

for z small with $T(\gamma_1)$ being described by 4-torus

$$\{(X_1, X_2, X_3, X_4) \in \mathbb{C}^4 |\ |X_1| = |X_2| = |X_3| = |X_4| = \epsilon\}$$

with $\epsilon \ll |X_0|$. We now use that the integral does not depend on the value of X_0 and insert a factor

$$1 = \frac{1}{2\pi i} \oint \mathrm{d}X_0 \frac{1}{X_0}$$

such that the integral becomes

$$\begin{aligned}
\Pi_1(z) =& \frac{1}{(2\pi i)^2} \oint \mathrm{d}X_0 \oint \mathrm{d}X_1 \oint \mathrm{d}X_2 \oint \mathrm{d}X_3 \oint \mathrm{d}X_4 \frac{1}{P_z(X)} \\
=& \frac{1}{(2\pi i)^2} \oint \mathrm{d}X_0 \oint \mathrm{d}X_1 \oint \mathrm{d}X_2 \oint \mathrm{d}X_3 \oint \mathrm{d}X_4 \frac{1}{X_0X_1X_2X_3X_4} \\
& \sum_{k=0}^\infty \left((X_0 + X_1 + X_2 + X_3 + X_4) \left(\frac{1}{X_0} + \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4}\right)\right)^k z^k.
\end{aligned}$$

The integrals will pick out the residue at the origin which we can easily evaluate since up to non-constant terms

$$\left((X_0+X_1+X_2+X_3+X_4)\left(\frac{1}{X_0}+\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}\right)\right)^k \sim \sum_{\sum k_i=k}\left(\frac{k!}{k_0!k_1!k_2!k_3!k_4!}\right)^2.$$

Thus, we conclude that the holomorphic period around $z=0$ is given by

$$\Pi_1(z)=(2\pi i)^3\sum_{k=0}^{\infty}\left(\sum_{\sum k_i=k}\left(\frac{k!}{k_0!k_1!k_2!k_3!k_4!}\right)^2\right)z^k.$$

To find the Picard-Fuchs equation, we search for a fourth order differential operator $\mathcal{L}$ annihilating the holomorphic period. Making the ansatz

$$\mathcal{L}=\sum_{k=0}^{4}P_k(z)\Theta^k$$

with polynomials P_k and $\Theta=z\frac{\mathrm{d}}{\mathrm{d}z}$, we find that

$$\begin{aligned}\mathcal{L}=&\Theta^4-z(35\Theta^4+70\Theta^3+63\Theta^2+28\Theta+5)+z^2(\Theta+1)^2(259\Theta^2+518\Theta+285)\\&-225z^3(\Theta+1)^2(\Theta+2)^2.\end{aligned}$$

A.9 Meromorphic Forms and Period Polynomials

We list the meromorphic forms and conventions for the period polynomials which we used for $\Gamma_0(14)$, $\Gamma_0(32)$, and $\Gamma_0(34)$ below. We checked the eigenvalues of the meromorphic forms under T_3 for at least 10000 terms in the q-expansion. The accuracy of the periods and quasiperiods has been checked with the Legendre relations which hold for at least 2000 decimal digits. For the computations, we used PARI/GP.

$\boldsymbol{\Gamma_0(14)}$

We consider the Hecke eigenforms

$$\begin{aligned}f_2&=\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)\in S_2^{\mathrm{new}}(\Gamma_0(14))\\f_4&=q-2q^2+8q^3+4q^4+\ldots\in S_4^{\mathrm{new}}(\Gamma_0(14)).\end{aligned}$$

We find that associated meromorphic forms $[F_2]\in\mathbb{S}_2(\Gamma_0(14))$ and $[F_4]\in\mathbb{S}_4(\Gamma_0(14))$ can be represented by

$$\begin{aligned}F_2&=\frac{G_2}{f_2^2}+\frac{181}{3}f_2\\F_4&=\frac{G_4}{f_2^4}+1267f_4\end{aligned}$$

with

$$\begin{aligned}G_2=&q-2q^2-3q^3+53q^4+107q^5-210q^6+49q^7+117q^8+\mathcal{O}(q^9)\in S_6(\Gamma_0(N))\\G_4=&8q^2-35q^3-4q^4+198q^5+734q^6+2062q^7+14424q^8+9873q^9+35118q^{10}\\&-56083q^{11}+27856q^{12}-182362q^{13}-51976q^{14}-368969q^{15}+83904q^{16}\\&-68498q^{17}+288580q^{18}+430179q^{19}+1741480q^{20}+\mathcal{O}(q^{21})\in S_{12}(\Gamma_0(N)).\end{aligned}$$

We define the periods and quasiperiods by

$$r_{f_k} = \omega_k^+ r_k^+ + \omega_k^- r_k^-$$
$$r_{F_k} = \eta_k^+ r_k^+ + \eta_k^- r_k^-$$

with $r_k^\pm$ given for a set of generators of $\Gamma_0(14)$ in Table A.1. The Legendre relations then read

$$\frac{1}{2\pi i}(\omega_2^+\eta_2^- - \omega_2^-\eta_2^+) = 2$$
$$\frac{1}{(2\pi i)^3}(\omega_4^+\eta_4^- - \omega_4^-\eta_4^+) = \frac{3528}{5}.$$

γ	$r_2^+(\gamma)$	$r_2^-(\gamma)$	$r_4^+(\gamma)$	$r_4^-(\gamma)$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	0	0	0	0
$\begin{pmatrix} 9 & -2 \\ 14 & -3 \end{pmatrix}$	1	1	$\tau^2 - \frac{13\tau}{42} + \frac{1}{42}$	$\tau^2 - \frac{3\tau}{7} + \frac{1}{21}$
$\begin{pmatrix} 41 & -11 \\ 56 & -15 \end{pmatrix}$	0	2	$-\frac{32\tau^2}{3} + \frac{40\tau}{7} - \frac{16}{21}$	$10\tau^2 - \frac{16\tau}{3} + \frac{5}{7}$
$\begin{pmatrix} 29 & -9 \\ 42 & -13 \end{pmatrix}$	0	2	$-6\tau^2 + \frac{26\tau}{7} - \frac{4}{7}$	$\frac{23\tau^2}{3} - \frac{14\tau}{3} + \frac{5}{7}$
$\begin{pmatrix} 11 & -4 \\ 14 & -5 \end{pmatrix}$	-1	1	$-\frac{2\tau^2}{3} + \frac{5\tau}{14} - \frac{1}{21}$	$\frac{\tau^2}{3} - \frac{5\tau}{21} + \frac{1}{21}$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	0	0	0	0

Table A.1: Values of period polynomials for generators of $\Gamma_0(14)$.

$\Gamma_0(32)$

We consider the Hecke eigenform

$$f = (\eta(4\tau)\eta(8\tau))^2 \in S_2^{\text{new}}(\Gamma_0(32)).$$

We find that an associated meromorphic form $[F] \in \mathbb{S}_2(\Gamma_0(32))$ can be represented by

$$F = \frac{G}{h} + 2f$$

with

$$h = \frac{\eta(32\tau)^{16}}{\eta(16\tau)^8} \in M_4(\Gamma_0(32))$$
$$G = q^{15} + \mathcal{O}(q^{19}) \in S_6(\Gamma_0(32)).$$

We define the periods and quasiperiods by

$$r_f = \omega^+ r^+ + \omega^- r^-$$
$$r_F = \eta^+ r^+ + \eta^- r^-$$

with the choice

$$r^+\left(\begin{pmatrix} 27 & -11 \\ 32 & -13 \end{pmatrix}\right) = r^-\left(\begin{pmatrix} 27 & -11 \\ 32 & -13 \end{pmatrix}\right) = 1.$$

The Legendre relations then read

$$\frac{1}{2\pi i}(\omega^+\eta^- - \omega^-\eta^+) = -\frac{1}{2}.$$

$\Gamma_0(34)$

We consider the Hecke eigenforms

$$\begin{aligned} f_2 &= q - q^2 + 2\sqrt{-2}q^3 + \mathcal{O}(q^4) \in S_2^{\text{new}}(\Gamma_0(34), \chi_{34}(33,\cdot)) \\ f_4 &= q - 2q^2 + 2\sqrt{-1}q^3 + \mathcal{O}(q^4) \in S_4^{\text{new}}(\Gamma_0(34), \chi_{34}(33,\cdot)). \end{aligned}$$

We find that associated meromorphic forms

$$\begin{aligned} [F_2] &\in \mathbb{S}_2(\Gamma_0(34), \chi_{34}(33,\cdot)) \\ [F_4] &\in \mathbb{S}_4(\Gamma_0(34), \chi_{34}(33,\cdot)) \end{aligned}$$

can be represented by

$$\begin{aligned} F_2 &= \frac{G_2}{h} + \alpha_2 f_2 \\ F_4 &= \frac{G_4}{h} + \alpha_4 f_4 \end{aligned}$$

with

$$\begin{aligned} h =& \frac{\eta(2\tau)^5\eta(17\tau)^{11}}{\eta(\tau)^3\eta(34\tau)^5} \in M_4(\Gamma_0(34)) \\ G_2 =& -153229813751q^{15} - 459689441253q^{16} + (3531532108992 - 1145079286032\sqrt{-2}+)q^{17} \\ &+ (878733286943 - 582046493424\sqrt{-2})q^{18} + (-419043845690 + 582046493424\sqrt{-2})q^{19} \\ &+ (-1363878385002 + 1284305025696\sqrt{-2})q^{20} + (-313817089303 + 166856185848\sqrt{-2})q^{21} \\ &+ \mathcal{O}(q^{22}) \in S_6(\Gamma_0(34), \chi_{34}(33,\cdot)) \\ G_4 =& (179835 - 20050\sqrt{-1})q^6 + (1491309 + 53250\sqrt{-1})q^7 + (-1046167 + 16944\sqrt{-1})q^8 \\ &+ (4189224 + 238802\sqrt{-1})q^9 + (-4766998 - 138514\sqrt{-1})q^{10} \\ &+ (18982341 + 708738\sqrt{-1})q^{11} + (-25952086 - 2061970\sqrt{-1})q^{12} \\ &+ (24204759 - 277294\sqrt{-1})q^{13} + (-51353479 - 5440012\sqrt{-1})q^{14} \\ &+ (97939573 - 1155540\sqrt{-1})q^{15} + (-118945037 - 13052668\sqrt{-1})q^{16} \\ &+ (97844466 - 9018588\sqrt{-1})q^{17} + (-326491115 - 28926796\sqrt{-1})q^{18} \\ &+ (150895777 - 9775724\sqrt{-1})q^{19} + (-501710900 - 42278510\sqrt{-1})q^{20} \\ &+ (-29394278 - 24284642\sqrt{-1})q^{21} + (-811045661 - 57899918\sqrt{-1})q^{22} \\ &+ (447002106 - 21771500\sqrt{-1})q^{23} + (-809721452 - 61744526\sqrt{-1})q^{24} \\ &+ (76325603 + 9834864\sqrt{-1})q^{25} + (-546262747 - 17599808\sqrt{-1})q^{26} \\ &+ (1320354189 + 99786514\sqrt{-1})q^{27} + (149488926 + 107252912\sqrt{-1})q^{28} \\ &+ (759963306 + 70310964\sqrt{-1})q^{29} + (-840148004 + 59824116\sqrt{-1})q^{30} \\ &+ (2698122690 + 238994596\sqrt{-1})q^{31} + \mathcal{O}(q^{32}) \in S_8(\Gamma_0(34), \chi_{34}(33,\cdot)) \end{aligned}$$

and

$$\begin{aligned} \alpha_2 =& \quad -(41392107 + 1442848\sqrt{-1} \pm (10047969 + 350256\sqrt{-1})\sqrt{17}) \\ \alpha_4 =& \frac{-11}{3264}(63552263336460 + 47553595658235\sqrt{-2} \\ &\pm (15942061865268 \pm 11752838193173\sqrt{-2})\sqrt{17}). \end{aligned}$$

The signs of $\sqrt{17}$ in α_2 and α_4 are chosen according to the point $z_\pm = 33 \pm 8\sqrt{17}$ for which we calculate the transition matrix. We again define the periods and quasiperiods by

$$\begin{aligned} r_{f_k} &= \omega_k^+ r_k^+ + \omega_k^- r_k^- \\ r_{F_k} &= \eta_k^+ r_k^+ + \eta_k^- r_k^- \end{aligned}$$

where we normalise so that

$$\begin{aligned}
r_2^+\left(\begin{pmatrix} -373 & 24 \\ -544 & 35 \end{pmatrix}\right) &= -\frac{9}{2} - \frac{9\sqrt{-2}}{4} \\
r_2^-\left(\begin{pmatrix} -373 & 24 \\ -544 & 35 \end{pmatrix}\right) &= \frac{9\sqrt{-2}}{2} \\
r_4^+\left(\begin{pmatrix} 103 & -20 \\ 170 & -33 \end{pmatrix}\right) &= \frac{502\tau^2}{85} - \frac{3314\tau}{1445} + \frac{322}{1445} + \sqrt{-1}\left(-\frac{154\tau^2}{85} + \frac{1018\tau}{1445} - \frac{99}{1445}\right) \\
r_4^-\left(\begin{pmatrix} -33 & 1 \\ -34 & 1 \end{pmatrix}\right) &= 2\tau^2 - \frac{2\tau}{17}.
\end{aligned}$$

Note that the rationality of the $r_k^\pm$ only holds for elements in $\Gamma_1(34) \subset \Gamma_0(34)$, i.e. when the Dirichlet character becomes trivial. As expected, we find that $r_2^\pm$ is then defined over $\mathbb{Q}(\sqrt{-2})$ and $r_4^\pm$ over $\mathbb{Q}(\sqrt{-1})$. The Legendre relations read

$$\begin{aligned}
\frac{1}{2\pi i}(\omega_2^+\eta_2^- - \omega_2^-\eta_2^+) &= -\frac{77\sqrt{17}}{44064}(88926424 + 7327582485\sqrt{-2}) \\
\frac{1}{(2\pi i)^3}(\omega_4^+\eta_4^- - \omega_4^-\eta_4^+) &= -\frac{\sqrt{17}}{20}(42217409 + 19534088\sqrt{-1}).
\end{aligned}$$

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