Weyl spinor conventions

Correcting a blunder on sheet 4 we will write (j_+, j_-) to denote representations of the Lorentz group, where $(\frac{1}{2}, 0)$ corresponds to a left (!) handed Weyl spinor. In particular this exchanges the transformation laws used on exercise sheet 4. (Note that this is only a list of conventions to avoid confusion and not meant to be a sufficient prerequisite for the exam)

- Lorentz algebra:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}) \tag{1}$$

Generators of rotations and boosts:

$$L^i = \frac{1}{2}\epsilon^{ijk}J_{jk}, \quad K^i = J^{0i} \quad \text{with} \quad \epsilon^{123} = 1 \tag{2}$$

Commutation relations:

$$[L^i, L^j] = i\epsilon^{ijm}L^m, \quad [L^i, K^j] = i\epsilon^{ijm}K^m \quad \text{and} \quad [K^i, K^j] = -i\epsilon^{ijk}L^k \tag{3}$$

General transformation on finite dimensional representation:

$$\Phi \to \exp\left(-i\mathbf{\Theta}\cdot\mathbf{L} - i\beta\cdot\mathbf{K}\right)\Phi \tag{4}$$

With the redefined generators $J_\pm = \frac{1}{2}(\mathbf{L} \pm i\mathbf{K})$ one gets commutation relations

$$[J_+^i, J_+^j] = i\epsilon^{ijk}J_+^k, \quad [J_-^i, J_-^j] = i\epsilon^{ijk}J_-^k \quad \text{and} \quad [J_+^i, J_-^j] = 0. \tag{5}$$

Representations of the Lorentz algebra are labelled by tuples $(j_+, j_-) \in \frac{1}{2}\mathbb{N} \otimes \frac{1}{2}\mathbb{N}$. Most important representations (for this course):

Representation	Name	Dimension
$(0, 0)$	complex scalar	1
$(\frac{1}{2}, 0)$	left handed Weyl spinor	2
$(0, \frac{1}{2})$	right handed Weyl spinor	2
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	Dirac spinor	4
$(\frac{1}{2}, \frac{1}{2})$	complex vector	4

Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

Transformation of left handed Weyl spinors:

$$\psi_L \to \Lambda_L\psi_L = \exp\left(-i\mathbf{\Theta}\cdot\frac{\sigma}{2} - \beta\cdot\frac{\sigma}{2}\right)\psi_L \tag{7}$$

Transformation of right handed Weyl spinors:

$$\psi_R \to \Lambda_R\psi_R = \exp\left(-i\mathbf{\Theta}\cdot\frac{\sigma}{2} + \beta\cdot\frac{\sigma}{2}\right)\psi_R \tag{8}$$