Physikalisches Institut
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Theoretische Physik

Exercise Sheet 1
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# Exercises Quantum Field Theory I

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–Presence exercises–

## 1 Classical Electrodynamics and Field Theory

In this exercise session we want to recall some basics in field theory. We focus on the simplest setup, which you should be familiar with from classical electrodynamics.

### 1.1 Warm-Ups

1. Recall the definition of the field strength tensor $F_{\mu\nu}$ from electrodynamics in terms of the electric and magnetic fields. How is it related to the potential $A_\mu$?

2. Which gauge transformation acting on $A_\mu$ leaves the theory invariant?

3. Consider a theory which is described by a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$, depending on a field $\phi$ and its derivative. Recall the expression for the equations of motion and furthermore recall the statement of Noether's theorem. How can you generalize the simplest field theoretical version of Noether's theorem to accomodate for

   - Multiple fields $\phi_i$?
   - Symmetries depending on multiple parameters,
     e.g. translations in $d > 1$ or rotations in $d > 2$?

   To which symmetry is the energy-momentum tensor associated and what is its general formula from Noether's theorem?

### 1.2 Exercises

1. Consider the following action for electrodynamics without sources

$$S = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

   and derive Maxwell's equations from that.

2. Calculate the energy-momentum tensor. Check wether it is symmetric!

3. In order to obtain a symmetric energy-momentum tensor, we can add a term of the form $\partial_\alpha K^{\alpha\mu\nu}$ (recall why?). Demanding that $\partial_\alpha K^{\alpha\mu\nu}$ is divergenceless, what does this imply for its symmetries? Use

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu, \tag{2}$$

to introduce the completed energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\alpha K^{\alpha\mu\nu}, \tag{3}$$

and derive the expressions for the electromagnetic energy and momentum densities.

4. How do you incorporate a source term? *Hint: How does a source term occur in Maxwell's equations? What modification is needed in the Lagrangian?* What would change in 1.-3. in this case?

## 2 The Lorentz Group I

In this exercise we would like to recall some basic facts about the Lorentz group, which you hopefully encountered earlier in your studies. We work with the convention, that the metric $g_{\mu\nu}$ has the following form $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and four-vectors are usually denoted by $x = (x^0, \vec{x})$. We sometimes write $\langle x, y\rangle = g_{\mu\nu} x^\mu y^\nu$. Furthermore the canonical basis of $\mathbb{R}^4$ is denoted by $e_\mu$ with 1 in the $\mu$-th component and 0 otherwise.

1. Remind yourself, what are space-, light- and time-like vectors!

2. What is the defining property of a Lorentz transformation?

3. How can one interpret spatial rotations as a Lorentz transformation?

4. Show explicitly, that the two transformations $T = \mathrm{diag}(-1, 1, 1, 1)$ and $S = \mathrm{diag}(1, -1, -1, -1)$ are elements of the Lorentz group. What is the interpretation of their action?

5. Following the definition, show that $|\Lambda^0_0| \geq 1$ and $\det\Lambda = \pm 1$. What is the meaning of $\Lambda^0_0 < 0$?

6. Show that the Lorentz transformations make up a group, called the Lorentz group $L = O(1,3)$, which has four disconnected branches (as a manifold)

$$\begin{aligned} &\mathcal{L}^\uparrow_+ : \Lambda^0_0 \geq +1, \quad \det\Lambda = +1, && \mathcal{L}^\uparrow_- : \Lambda^0_0 \geq +1, \quad \det\Lambda = -1 \\ &\mathcal{L}^\downarrow_- : \Lambda^0_0 \leq -1, \quad \det\Lambda = -1, && \mathcal{L}^\downarrow_+ : \Lambda^0_0 \leq -1, \quad \det\Lambda = +1. \end{aligned}$$

   Which component is connected to the identity? For the case that $\Lambda^0_0 \geq 1$, $\Lambda$ is called orthochronous. If $\det\Lambda = 1$, then $\Lambda$ is called proper. The proper Lorentz transformations $L_+ := \mathcal{L}^\uparrow_+ \cup \mathcal{L}^\downarrow_+$ are called $SO(1,3)$. How can one obtain the other branches of the Lorentz group from the orthochronous, proper branch?

7. Consider the set of skew-symmetric matrices $\hat{L}$, i.e. $\langle x, Ay\rangle = -\langle Ax, y\rangle$ w.r.t. the Minkowski metric. Define furthermore

$$\omega_{\mu\nu} x = e_\mu \langle e_\nu, x\rangle - e_\nu \langle e_\mu, x\rangle, \quad \mu < \nu. \tag{4}$$

   Show that $\hat{L}$ is a six-dimensional vector space with basis $\omega_{\mu\nu}$ and $[A, B] = AB - BA \in \hat{L}$ for $A, B \in \hat{L}$.

8. Calculate $[\omega_{\mu\nu}, \omega_{\kappa\lambda}]$!
9. In addition, show that for $A \in \hat{L}$

$$\Lambda(\tau) = \exp(\tau A) \in L_+, \tag{5}$$

*Hint: You might want to use that* $\det \exp(A) = \exp(\mathrm{Tr} A)$.

This promotes the vector space $\hat{L}$ to a Lie algebra of the Lie group $L_+$ and $\omega_{\mu\nu}$ are called the generators of $L_+$ (why only $L_+$?). *Note: The Lie algebra is the vector space of infinitesimal transformations with the Lie bracket* $[\cdot,\cdot]$ *as an inner product.*

–Homework–

## 3 The complex scalar field (10 pts.)

Let $\phi : \mathbb{R}^4 \to \mathbb{C}$ be a complex scalar field that obeys the Klein-Gordon equation. The action is given by

$$S = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi). \tag{6}$$

This theory is best analyzed by considering $\phi(x)$ and $\phi^*(x)$ as the basic dynamic variables.

1. Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and write down the canonical commutation relations. Show that the Hamiltonian is given by

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi). \tag{7}$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation. **3 pts.**

2. Expand the field operators into fourier modes and express $H$ in terms of creation and annihilation operators. Show that the theory contains two sets of particles of mass $m$. **2 pts.**

3. Show that the theory is invariant under $\phi(x) \mapsto e^{i\alpha}\phi(x)$ and therefore exhibits a global $U(1)$ symmetry. *Note: Global means that* $\alpha$ *does not depend on* $x$. **1 pt.**

4. Rewrite the conserved charge

$$Q = \int d^3x \frac{i}{2} (\phi^* \pi^* - \pi \phi), \tag{8}$$

in terms of creation and annihilation operators and evaluate the charge of the particles of each type. **2 pts.**

5. Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_\alpha(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of the part above, and the other three given by

$$Q^i = \int d^3x \frac{i}{2} (\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b), \tag{9}$$

where $\sigma^i$ are the Pauli matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$)! **2 pt.**