Physikalisches Institut
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Theoretische Physik

Exercise Sheet 10
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SS 15

Exercises Quantum Field Theory I

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http://www.th.physik.uni-bonn.de/klemm/qft1ss15/

–Homework–

1 Bhabha scattering (15 pts.)

The process $e^-e^+ \to e^-e^+$ called Bhabha scattering is important to calibrate electron-positron colliders and in particular responsible for most of the background.

1. Draw the two diagrams contributing to the invariant matrix element at tree-level. Denote the momenta of the incoming particles by p_1, p_2 and of the outgoing particles by k_1, k_2. Write down the corresponding expression for $\mathcal{M}$ and be careful about the relative sign of the two terms. *Hint: You can sketch the Wick-contractions and the necessary reorderings of the fermions to make sure.* **3 pts.**

2. Prove the following identities that you will need to evaluate $|\mathcal{M}|^2$ and several traces that occur:

   - To obtain the absolute value squared, prove that

$$\begin{aligned}(\bar{v}(p)\gamma^\mu u(k))^* =&\bar{u}(k)\gamma^\mu v(p),\\ (\bar{v}(p)\gamma^\mu v(k))^* =&\bar{v}(k)\gamma^\mu v(p),\\ (\bar{u}(p)\gamma^\mu u(k))^* =&\bar{u}(k)\gamma^\mu u(p).\end{aligned} \tag{1}$$

   - To evaluate spin summed Dirac bilinears prove the relations

$$\begin{aligned}\sum_{s,s'} \bar{u}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu u^{s'}(p') &= \mathrm{Tr}\left[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu\right],\\ \sum_{s,s'} \bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu v^{s'}(p') &= \mathrm{Tr}\left[(\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\right],\\ \sum_{s,s'} \bar{u}^{s'}(p')\gamma^\mu v^s(p)\bar{v}^s(p)\gamma^\nu u^{s'}(p') &= \mathrm{Tr}\left[(\not{p}' + m)\gamma^\mu(\not{p} - m)\gamma^\nu\right],\\ \sum_{s,s'} \bar{v}^{s'}(p')\gamma^\mu v^s(p)\bar{v}^s(p)\gamma^\nu v^{s'}(p') &= \mathrm{Tr}\left[(\not{p}' - m)\gamma^\mu(\not{p} - m)\gamma^\nu\right],\end{aligned} \tag{2}$$

   using the completeness relations for Dirac spinors.

- Show the following trace relations for products of gamma matrices:

$$\begin{aligned} \mathrm{Tr}(\mathbf{1}) =& 4 \\ \mathrm{Tr}(\text{odd \# of } \gamma\text{'s}) =& 0 \\ \mathrm{Tr}(\gamma^\mu\gamma^\nu) =& 4\eta^{\mu\nu} \\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) =& 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \\ \mathrm{Tr}(\gamma^5) =& 0 \\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) =& 0 \end{aligned} \tag{3}$$

Note: For many of these formula it is appropriate to calculate just one case and quickly describe what changes for the rest and why. **4 pts.**

3. In most experiments the particle beam is unpolarized and also the detector is usually not sensitive to polarization. In this setup one has to average over the incoming particle spins and sum over the final state spins. This means we want to compute

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'} \left|\mathcal{M}(s,s' \to r,r')\right|^2. \tag{4}$$

Introduce mandelstam variables[1]

$$\begin{aligned} s =& (p_1+p_2)^2 = 2(m^2 + p_1\cdot p_2) = 2(m^2 + k_1\cdot k_2), \\ t =& (p_1-k_1)^2 = 2(m^2 - p_1\cdot k_1) = 2(m^2 - p_2\cdot k_2), \\ u =& (p_1-k_2)^2 = 2(m^2 - p_1\cdot k_2) = 2(m^2 - k_1\cdot p_2) \end{aligned} \tag{5}$$

and use the identities proven in the previous part of the exercise to calculate (4). *Hint: You can use*

$$\begin{aligned} &Tr\left\{(\not{k}_2 - m)\gamma^\mu(\not{p}_2 - m)\gamma_\nu(\not{p}_1 + m)\gamma_\mu(\not{k}_1 + m)\gamma^\nu\right\} \\ =& -16(m^2 k_1\cdot k_2 + 2k_1\cdot p_2\, k_2\cdot p_1 - m^2 k_1\cdot p_1 + m^2 k_1\cdot p_2 \\ &+ m^2 k_2\cdot p_1 - m^2 k_2\cdot p_2 + 2m^4 + m^2 p_1\cdot p_2). \end{aligned} \tag{6}$$

You obtain **3 bonus points** if you carry out the calculation without setting the mass to zero, but it is sufficient to go to the high energy limit[2] where $m \sim 0$. In the massless limit the result should be

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = 2e^4\left[\frac{s^2+u^2}{t^2} + \frac{t^2+u^2}{s^2} + \frac{2u^2}{st}\right]. \tag{7}$$

6 pts.

4. Now, if you haven't done that before, take the massless limit and calculate the differential cross section

$$\left(\frac{d\sigma}{d\cos\theta}\right)_{CM} = \frac{|\mathcal{M}|^2}{32\pi E_{\text{cm}}^2}. \tag{8}$$

Replace the Mandelstam variables by the center of mass energy, $\cos(\theta)$ and $\sin(\theta)$ where θ denotes the angle between $\mathbf{p}_1$ and $\mathbf{k}_1$. **2 pts.**

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[1] Note that the relations on the right hand side are only valid because all particles share the same mass.

[2] We assume that the energy is still smaller than the mass of the Z boson $m_z \approx 91\,\text{GeV}$, so that we can neglect corrections from the weak interaction.