Physikalisches Institut
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Theoretische Physik

Exercise Sheet 12 (the last one)
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SS 15

Exercises Quantum Field Theory I

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http://www.th.physik.uni-bonn.de/klemm/qft1ss15/

–Homework–

1 Dimensional regularization of ϕ^4 theory at 1-loop (15 pts.)

Until now all the processes you calculated in the homework exercises were at tree-level. This means that all momenta of the internal particles were fixed by momentum conservation. Higher order contributions come from loop diagrams, where some internal momenta are unconstrained and lead to integrals which in general are divergent. Two steps are necessary to still get a physical answer out of these formal expressions. First one has to regularize the integral. This means that one introduces a parameter for which the expression diverges at a specific value and then evaluates the integral. The result is an expression which has a pole or diverges logarithmically for the physical value of the auxilliary parameter. As a second step one renormalizes the theory, which roughly means that the divergent contribution gets an interpretation in terms of Feynman rules and can be absorbed in the coupling constants. One direct consequence of this procedure, which we will however not treat on this sheet, is that in general the physical values of the coupling constants depend on the energy. In this exercise we will only carry out the first step and regularize the one-loop divergences in ϕ^4 theory. The regularization scheme we will use is called *dimensional regularization* and depends on the at first rather weird technique of making the dimension of space-time infinitesimally smaller than four.[1]

1. We assume that the integral

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - A)^n} \tag{1}$$

is analytic in p^0 and therefore the evaluation can be performed in euclidean space as well. This is achieved by substituting $p^0 \to ip^0$ and called *Wick rotation.* Do the substitution. *Hint: To feel more comfortable with this seemingly radical step just imagine tilting the integration contour used to derive the Fenyman propagator gently by ninety degrees.* **0.5 pts**

2. Prove the identity

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - A)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} A^{d/2-n} \tag{2}$$

[1] A very short discussion of the mathematical justification can be found for example in http://einrichtungen.ph.tum.de/T30f/lec/QFT2/RevModPhys.47.849.pdf.

false using polar coordinates in d dimensions. *Hint: The measure is given by*

$$d^d p = P^{d-1} dP d\phi \sin\theta_1 d\theta_1 \sin^2\theta_2 d\theta_2 \dots \sin^{d-2}\theta_{d-2} d\theta_{d-2}, \tag{3}$$

where the variables take values in the intervals

$$0 \le P \le \infty, \quad 0 \le \phi < 2\pi, \quad 0 \le \theta_i < \pi \quad \textit{with} \quad i = 1, \dots, d-2. \tag{4}$$

Use the formula

$$\int_0^{\pi/2} (\sin t)^{2x-1} (\cos t)^{2y-1} dt = \frac{1}{2} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \textit{for} \quad Re(x), Re(y) > 0 \tag{5}$$

and the representation of the beta function

$$\int dx\, x^a (1-x)^b = B(a+1, b+1) = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}. \tag{6}$$

Or better

$$B(a+1, b+1) = \int_0^\infty t^a (t+1)^{-2-a-b}, \quad \textit{for} \quad Re(x) > 0,\ Re(y) > 0. \tag{7}$$

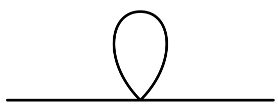
4 pts

3. For convenience we will seperate the energy dimension from the coupling constant and define the action of our theory in 2ω dimensions as

$$S_\omega[\phi] = \int d^{2\omega} x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} (\mu^2)^{2-\omega} \phi^4 \right], \tag{8}$$

where we introduced some arbitrary constant μ^2 with the dimension of mass. What is the Feynman rule for the ϕ^4-vertex? **0.5 pts**

4. Evaluate the tadpole diagram



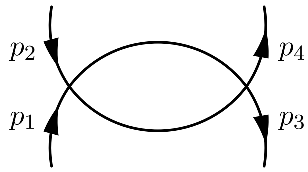
in 2ω dimensions and expand the gamma function and the term $(\dots)^{2-\omega}$ neglecting terms of order $\mathcal{O}(\omega - 2)$ or higher. *Hint: You can use the expansion*

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon) \right], \tag{9}$$

where γ is the Euler-Mascheroni constant and $\psi(x)$ is the Digamma function. Note that you have to substitute $\int \frac{d^4 p}{(2\pi)^4} \to \int \frac{d^{2\omega} p}{(2\pi)^{2\omega}}$ compared to your usual formulae. ***2 pts.***

5. Did you miss a symmetry factor?

6. A more complicated example is given by the "fish":



In the integrand you will have a product of two momentum space propagators. Use the formula

$$\frac{1}{A \cdot B} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2}, \tag{10}$$

commute the integral signs and, assuming convergence, shift the variable to bring your integral into a form to which formula (2) applies. **5 pts.**

7. Expand to the same order as before and use

$$\int_0^1 dx \ln\left[1 + \frac{4}{a}x(1-x)\right] = -2 + \sqrt{1+a}\ln\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right), \quad \text{for} \quad a > 0 \tag{11}$$

to evaluate the remaining integral and obtain

$$(\mu^2)^{2-\omega}\frac{\lambda^2}{32\pi^2}\left[\frac{1}{2-\omega} + \psi(1) + 2 + \ln\frac{4\pi\mu^2}{m^2} - \sqrt{1+\frac{4m^2}{p^2}}\ln\left\{\frac{\sqrt{1+\frac{4m^2}{p^2}}+1}{\sqrt{1+\frac{4m^2}{p^2}}-1}\right\} + \mathcal{O}(2-\omega)\right],$$

with $p = p_1 + p_2$ being the sum of the two incoming momenta. *Note: In order to obtain the t- and u-channel diagrams just substitute $p^2 = s \to t, u$ respectively.* **3 pts.**