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Exercises Quantum Field Theory I

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`http://www.th.physik.uni-bonn.de/klemm/qft1ss15/`

–Presence exercises–

In this exercise session we want to recall some basic notions from complex analysis and especially on integration in the complex plane. This becomes important in the evaluation of propagators which are a central object in QFT.

1 Some basics on integration in the complex plane

This exercise is a short recap of some facts about complex integration, that we need in the next exercise.

1. Integrate the function $f(z) = z^n$, $n \in \mathbb{Z}$ along the two closed paths in the complex plane parametrised by
$$\gamma(t) = re^{\pm it},\ 0 \leq t \leq 2\pi \tag{1}$$
for $n = -1$ and $n \neq -1$. Do the results depend on the parameter r? Do they depend on the orientation of the path (i.e. the $\pm$ sign)?

2. Relate this result to the existence or non-existence of an antiderivative $F(z)$, $\partial_z F(z) = f(z)$ along the complete path(s).

3. The Laurent series is a generalisation of the Taylor series that includes poles/singularities,
$$f(z) = \sum_{m=-\infty}^{\infty} a_m (z - z_0)^m. \tag{2}$$
The coefficient of the single poles is called the residue of f in z_0,
$$a_{-1} = \mathrm{res}_{z_0} f. \tag{3}$$
In the following exercise (and quantum field theory in general) you will need the residue theorem:

Residue theorem

Let γ be a closed path in $U \subset \mathbb{C}$, S a discrete set, $f(z)$ holomorphic in $U \setminus S$ and γ meets none of the points in S. Then
$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in S} n(\gamma, a) \mathrm{res}_a f, \tag{4}$$
where $n(\gamma, a)$ is the winding number of the path γ with respect to a.

As an example show that

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}, \tag{5}$$

by transforming the limits of integration to $(-\infty, \infty)$ and closing the contour using an arc $\gamma' := \{x = Re^{it},\, 0 \le t \le \pi\}$.

2 Properties of the propagator $\Delta(x-y)$

Expand the real scalar field $\phi(x)$ in annihilation and creation operators, calculate $\Delta(x-y) = -i\langle 0|\phi(x)\phi(y)|0\rangle$ and show that it has the following properties:

1. $\Delta(x-y)$ is a Lorentz-invariant function,
2. $\Delta(x-y) = -\Delta(y-x)$,
3. $\Delta(x-y)$ obeys the following boundary conditions:

$$\Delta(0, \vec{x}-\vec{y}) = 0, \; \frac{\partial}{\partial x^0}\Delta(x^0-y^0, \vec{x}-\vec{y})\bigg|_{x^0=y^0} = -\delta^{(3)}(\vec{x}-\vec{y}). \tag{6}$$

4. $\Delta(x-y)$ obeys the homogeneous Klein-Gordon equation

$$(\Box + m^2)\Delta(x-y) = 0. \tag{7}$$

5. $\Delta(x-y)$ vanishes for spacelike arguments:

$$(x-y)^2 < 0 \rightarrow \Delta(x-y) = 0. \tag{8}$$

–Homework–

3 Poincaré algebra (10 pts.)

Perhaps the main guiding principle of quantum field theory is that of symmetry. In particular all the fields have to fall into representations of the symmetries that leave a given theory invariant. Even if internal symmetries such as gauge or flavour symmetries are absent the particles arising as excitations in a relativistic quantum field theory are classified by irreducible representations of the Poincaré group. The Poincaré group as well as the group of rotations and many other groups that one encounters in modern physics are examples of Lie groups. This means that they are continuous and have the geometrical structure of a manifold of a certain dimension. Elements connected to the identity can be obtained by exponentiating generators of infinitesimal transformations. The vector space spanned by the latter is called Lie algebra and encodes many properties of the group.

1. What must $m^\mu{}_\nu$ satisfy so that $\Lambda^\mu{}_\nu = \delta^\mu_\nu + i\epsilon m^\mu{}_\nu + \mathcal{O}(\epsilon^2)$ is an infinitesimal Lorentz transformation? What is the constraint on $t^i{}_j$ for an ordinary infinitesimal rotation $R^i{}_j = \delta^i_j + i\epsilon t^i{}_j + \mathcal{O}(\epsilon^2)$? **1 pts.**

2. As generators for the Lorentz group in D dimensional space-time one can take the infinitesimal „rotations“ in the $\frac{1}{2}D(D-1)$ coordinate planes $[\rho\sigma]$ given by
$$m_{[\rho\sigma]}{}^{\mu}{}_{\nu} = i\left(\delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\rho\nu}\right) = -m_{[\sigma\rho]}{}^{\mu}{}_{\nu}. \tag{9}$$
The transformation acts on functions according to
$$\hat{R}f(x) = f(\hat{R}^{-1}x), \tag{10}$$
and from the Taylor expansion
$$f\left(x^{\mu} - \epsilon^{i}\frac{\partial x'^{\mu}}{\partial a^{i}}\bigg|_{a=0}\right) = f(x) - \epsilon^{i}\frac{\partial x'^{\mu}}{\partial a^{i}}\frac{\partial}{\partial x^{\mu}}f(x) + \mathcal{O}(\epsilon^{2}), \tag{11}$$
where a^{i} parametrises the transformation, you find that the generators of the representation on functions can be calculated using
$$L_{i} = i\frac{\partial x'^{\mu}}{\partial a^{i}}\frac{\partial}{\partial x^{\mu}}. \tag{12}$$
Note: the index i can in general stand for multiple and for example Lorentz indices.
Calculate the corresponding generators of the Lorentz group $L_{[\rho\sigma]} \equiv L_{\rho\sigma}$ and show the commutation relations
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\sigma}L_{\nu\rho}). \tag{13}$$
2 pts.

3. The Poincaré group can be obtained by allowing translations $x^{\mu} \to x^{\mu} + a^{\mu}$ in addition to the rotations and boosts of the Lorentz group. Calculate the differential operator representation of the generator of translations P_{μ}. Calculate the commutators $[P_{\mu}, P_{\nu}]$ and $[L_{\mu\nu}, P_{\lambda}]$. Does the algebra close? **2 pts.**

4. The Pauli-Lubanski four-vector is defined in terms of P_{μ} and $L_{\mu\nu}$ as $W^{\lambda} = \frac{1}{2}\epsilon^{\lambda\sigma\mu\nu}L_{\mu\nu}P_{\sigma}$. Prove the commutation relations
$$[W^{\lambda}, L^{\mu\nu}] = i(W^{\nu}\eta^{\mu\lambda} - W^{\mu}\eta^{\nu\lambda}), \quad [W^{\lambda}, P^{\sigma}] = 0, \quad [W^{\lambda}, W^{\sigma}] = i\epsilon^{\lambda\sigma\mu\nu}W_{\nu}P_{\mu}. \tag{14}$$
Hint: for the first commutator use the identity
$$\eta^{\delta\nu}\epsilon^{\rho\sigma\lambda\mu} + \eta^{\rho\nu}\epsilon^{\sigma\lambda\mu\delta} + \eta^{\sigma\nu}\epsilon^{\lambda\mu\delta\rho} + \eta^{\lambda\nu}\epsilon^{\mu\delta\rho\sigma} + \eta^{\mu\nu}\epsilon^{\delta\rho\sigma\lambda} = 0. \tag{15}$$
3 pts.

5. Irreducible representations can be classified by the eigenvalues of the Casimir operators. These are operators that commute with all other generators of the group. Show that $P^{\mu}P_{\mu}$ and $W^{\mu}W_{\mu}$ are Casimir operators of the Poincaré group. **2 pts.**

In fact $P^{\mu}P_{\mu}$ and $W^{\mu}W_{\mu}$ are the only Casimir operators of the Poincaré group and we can use them to classify the representations. There are six cases to consider
$$\begin{array}{rl} (i) & p^{2} = m^{2} > 0, \quad p^{0} > 0, \\ (ii) & p^{2} = m^{2} > 0, \quad p^{0} < 0, \\ (iii) & p^{2} = m^{2} = 0, \quad p^{0} > 0, \\ (iv) & p^{2} = m^{2} = 0, \quad p^{0} < 0, \\ (v) & p^{\mu} = 0, \\ (vi) & p^{2} < 0. \end{array} \tag{16}$$

The first and third classes correspond to physical massive and massless particles, the fifth class is the vacuum and the sixth should correspond to virtual particles. The other classes are probably unphysical. (See *Lewis H. Ryder, Quantum Field Theory, p.60f.* Actually reading p. 55-64 is highly recommended.) For the case of a physical massive particle the eigenvalues to $W^\mu W_\mu$ are of the form $W^\mu W_\mu = -m^2 s(s+1)$ and classify the spin of the particle. For massless particles W^μ is proportional to P^μ and the proportionality constant λ classifies the helicity. For example the helicity of a photon is $\lambda = \pm 1$.