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Theoretische Physik

Exercise Sheet 3
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SS 15

Exercises Quantum Field Theory I

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http://www.th.physik.uni-bonn.de/klemm/qft1ss15/

–Presence exercises–

1 The energy of the vacuum

In the lecture you saw that the Hamiltonian of the real Klein-Gordon field is given by

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right), \tag{1}$$

which after inserting the expansions of ϕ and π in creation and annihilation operators becomes

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right) = \int \frac{d^3p}{(2\pi)^3}\omega_p \left(a_p^\dagger a_p + \frac{1}{2}(2\pi)^3\delta^{(3)}(0) \right). \tag{2}$$

While the first term is just the number operator of particles with momentum $\vec{p}$ multiplied with the energy of a single particle, the second term shifts the energy and would be interpreted as the energy of the vacuum

$$E_0 = \int d^3p \, \frac{\omega_p}{2} \delta^{(3)}(0). \tag{3}$$

It is however infinite due to the delta distribution and the divergence of the integral.

1. Show that the infinity due to $\delta^{(3)}(0)$ arises because we are looking at the energy of an infinite Volume.
2. The energy density
$$\mathcal{E}_0 = \frac{E_0}{V} = \int d^3p \, \frac{\omega_p}{2} \tag{4}$$
still diverges. What could be the reason?

In a non-gravitational theory we are only interested in energy differences so we can drop the infinite contribution and obtain meaningful results.[1] The absolute value of the energy density of the vacuum reappears in cosmology as the cosmological constant Λ which is measured to be small but non-zero. It is one of the major open questions how this result can be explained from microscopic degrees of freedom.

[1] The prediction of quantum electrodynamics for the anomalous magnetic dipole moment of the electron agrees with the experimentally measured value to more than 10 significant figures.

2 The Casimir Effect

When you insert two parallel, perfectly conducting plates into the vacuum the boundary conditions on the electromagnetic field demand the orthogonal component of the wave-vector to be quantised. This leads to a shift in the vacuum energy depending on the distance between the plates which therefore experience a force. The effect of the quantised momenta can be studied analogously for the scalar field.

1. Consider the real, massless scalar field in a periodic $d = 1 + 1$ dimensional space-time, i.e. $\phi(x + L) = \phi(x)$ and insert two „conducting" plates with distance d. What is the energy $E(L, d)$ of the vacuum?

2. In the electromagnetic case the screening mechanism is due to freely moving electrons. It is therefore natural to assume that the screening breaks down for very high frequencies. Suppress the high frequency modes in the energy $E(L, d) = \sum_{n=1}^{\infty} c_n$ by introducing a regulator $c_n \to c_n e^{-an\pi/d}$ and expand in the cut-off distance $a \ll d$.

3. Calculate the force acting on the plates. *Note: it can be shown that the result is independent of the choice of regulator. For example, a sharp cut-off would give the same result.*