Physikalisches Institut
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Theoretische Physik

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Exercises Quantum Field Theory I

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`http://www.th.physik.uni-bonn.de/klemm/qft1ss15/`

–Homework–

1 Lorentz algebra II (15 pts.)

For a quantum field theory to be Lorentz invariant it is sufficient that the Lagrangian density $\mathcal{L}$ transforms as a scalar field, i.e. $\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(\Lambda^{-1}x)$. With the density of the Klein-Gordon field you have already seen one example where the field itself is scalar and corresponding particles have spin $s = 0$. Note that a scalar field transforms in an infinite dimensional representation of the Lorentz group with the generators represented by differential operators as was proven on sheet 2.[1] More general fields Φ^i can be obtained by allowing for internal degrees of freedom which transform in a finite dimensional representation, i.e.

$$\Phi(x)^i \to R(\Lambda)^i{}_j \Phi^j(\Lambda^{-1}x) \tag{1}$$

and quantum excitations of these fields lead to particles of spin $s = \frac{1}{2}, 1, \frac{3}{2}, 2$. In this exercise we want to classify the finite dimensional representations of the Lorentz algebra.[2]

1. Remember that the Lie algebra of the Lorentz group spanned by the generators $J^{\mu\nu}$ satisfies the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{2}$$

Note that the algebra is independent of the representation and for the internal degrees of freedom we are interested in finite dimensional representations so right now it is better to think about the generators as anti-symmetric matrices instead of differential operators. Define the generators of rotations and boosts as

$$L^i = \frac{1}{2}\epsilon^{ijk}J_{jk}, \qquad K^i = J^{0i}, \tag{3}$$

where $i, j, k = 1, 2, 3$. A Lorentz transformation[3] can then be written

$$\Phi \to \exp\left(-i\mathbf{\Theta}\cdot\mathbf{L} - i\beta\cdot\mathbf{K}\right)\Phi. \tag{4}$$

Show in the representation on 4-vectors, i.e. with the generators given as (9) on sheet 2, that a transformation with $\mathbf{\Theta} = (0, 0, \Theta)$ and $\beta = \vec{0}$ leads to a counterclockwise rotation in the xy-plane. Show that $\mathbf{\Theta} = \vec{0}$ and $\beta = (0, 0, \eta)$ leads to a boost in z-direction. **3 pts.**

[1] The number of (not necessarily independent) generators is the same for all representations but the space on which the generators act, in this case the space of functions, is infinite dimensional.

[2] Representations of the Lie algebra correspond to representations of the connected component of the Lie group. Transformations not in the proper orthochronous branch of the Lorentz group can mix representations of the Lorentz algebra. One example are left- and right-handed Weyl spinors which are interchanged under parity transformations.

[3] By Lorentz transformation we generally mean proper orthochronous.

2. Write the commutation relations among these re-defined generators in a concise form (i.e. *not* by inserting all possible indices). Show that the combinations

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}) \quad \text{and} \quad \mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K}) \tag{5}$$

commute with one another and separately satisfy the commutation relations of angular momentum. **3 pts.**

This shows that the (complexified) lie algebra of the Lorentz group is isomorphic to the (complexified) lie algebra of two independent copies of $SU(2)$. The representation theory of $SU(2)$ should already be familiar to you. Representations are labelled with half-integers $j \in \frac{1}{2}\mathbb{N}$ and the dimension for a given j is $d = 2j+1$. In fact finite dimensional representations of the Lorentz group can be labelled by a tuple $(j_+, j_-) \in \frac{1}{2}\mathbb{N} \otimes \frac{1}{2}\mathbb{N}$ and the dimension is $(2j_- + 1)(2j_+ + 1)$.

3. Lets take a closer look at the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. In $(\frac{1}{2}, 0)$ the generators ${J_+}^i = \frac{\sigma^i}{2}$ are represented by the pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

while $\mathbf{J}_- = 0$. What is the dimension of this representation? An object in this representation is called a left-handed Weyl spinor usually denoted ψ_L. What is the action

$$\psi_L \to \Lambda_L \psi_L \tag{7}$$

of a general Lorentz transformation on ψ_L in terms of rotations $\boldsymbol{\Theta}$ and boosts β? An object in the $(0, \frac{1}{2})$ representation is called a right-handed Weyl spinor. How does the Lorentz group act on ψ_R? **1 pt.**

4. Show that $\sigma^2 \sigma^i \sigma^2 = -\sigma^{i*}$. Use this to show that $\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R$ and $\psi_L^c = i\sigma^2 \psi_L^*$ transforms as a right-handed Weyl spinor, i.e.

$$\psi_L^c \to \Lambda_R \psi_L^c, \tag{8}$$

while on the other hand $\psi_R^c = -i\sigma^2 \psi_R^*$ transforms as a left-handed Weyl spinor, i.e.

$$\psi_R^c \to \Lambda_L \psi_R^c. \tag{9}$$

2 pt.

To show that $(\frac{1}{2}, \frac{1}{2})$ corresponds to the vector representation first observe that a general 2×2 hermitian matrix can be parametrised as

$$\mathbf{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{10}$$

5. Show that $\det \mathbf{x} = x^\mu \eta_{\mu\nu} x^\nu$. Now introduce two bases of hermitian 2×2 matrices

$$\sigma^\mu = (\mathbf{1}, \sigma^i), \quad \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i), \tag{11}$$

and show that

$$\begin{aligned} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu &= 2\eta^{\mu\nu}\mathbf{1}, \\ \text{tr}(\sigma^\mu \bar{\sigma}_\nu) &= 2{\delta^\mu}_\nu. \end{aligned} \tag{12}$$

Finally show

$$\mathbf{x} = \bar{\sigma}_\mu x^\mu, \quad x^\mu = \frac{1}{2}\text{tr}(\sigma^\mu \mathbf{x}). \tag{13}$$

Thus, there is a one to one correspondence between 4-vectors and hermitian 2×2 matrices where the Minkowski norm corresponds to the determinant. **1 pts.**

6. A linear map of hermitian matrices $\mathbf{x} \to \mathbf{x}' \equiv A\mathbf{x}A^\dagger$ preserves the determinant if $|\det A| = 1$ and since a phase cancels in the transformation we can take $A \in SL(2, \mathbb{C})$. The linear transformation $x'^\mu = \Lambda(A)^\mu{}_\nu x^\nu$ on the associated 4-vector is norm preserving and therefore a Lorentz transformation. Using the previous results it is given by

$$\Lambda(A)^\mu{}_\nu = \frac{1}{2}\text{tr}(\sigma^\mu A \bar{\sigma}_\nu A^\dagger). \tag{14}$$

   Show that $A\bar{\sigma}_\mu A^\dagger = \bar{\sigma}_\nu \Lambda^\nu{}_\mu$ and $A^\dagger \sigma_\mu A = \sigma_\nu (\Lambda^{-1})^\nu{}_\mu$. **2 pts.**

7. An element in the $(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ representation can be specified by giving a left- and a right-handed Weyl spinor ψ_L, ξ_R. Define $\xi_L = -i\sigma^2 \xi_R^*, \psi_R = i\sigma^2 \psi_L^*$ and calculate explicitly that the bilinear $\xi_R^\dagger \sigma^\mu \psi_R$ transforms under a boost in z-direction as a 4-vector. **2 pts.**

Note that $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$ also transforms as a 4-vector and both bilinears are in general complex. To obtain the real vector representation one has to demand the (Lorentz invariant) reality condition $\xi_R^\dagger \sigma^\mu \psi_R = \xi_L^\dagger \bar{\sigma}^\mu \psi_L$.

8. Show that $(\xi_R^\dagger \sigma^\mu \psi_R)^* = \xi_L^\dagger \bar{\sigma}^\mu \psi_L$. **1 pt.**