Physikalisches Institut
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Theoretische Physik

Exercise Sheet 5
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SS 15

Exercises Quantum Field Theory I

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http://www.th.physik.uni-bonn.de/klemm/qft1ss15/

Note that there was a blunder on sheet 4 where the tuples should have been (j_+, j_-) and the generators of the left handed Weyl representations are $J_+ = \frac{\sigma^i}{2}$, $J_- = 0$.

–Homework–

1 The Dirac representation (7 pts.)

The finite dimensional representations of the Lorentz algebra are classified by tuples $(j_+, j_-) \in \frac{1}{2}\mathbb{N} \otimes \frac{1}{2}\mathbb{N}$ and you showed in the last homework that left- and right handed Weyl spinors, which are in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation respectively, transform as

$$\psi_L \to \exp\left(-i\Theta \cdot \frac{\sigma}{2} - \beta \cdot \frac{\sigma}{2}\right)\psi_L \equiv \Lambda_L \psi, \quad \psi_R \to \exp\left(-i\Theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2}\right)\psi_R \equiv \Lambda_R \psi_R. \tag{1}$$

Remember that apart from the proper orthochronous branch the full Lorentz group has three more components not connected to the identity, one of which can be obtained by a space inversion or parity transformation, which in the 4-vector representation looks like

$$P^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

It acts on the generators as

$$J^i \to J^i, \quad K^i \to -K^i \tag{3}$$

and therefore exchanges $j_- \leftrightarrow j_+$. In order to construct a manifestly parity invariant theory with spinors (like quantum electrodynamics) one can work with the Dirac representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. An object in this representation can be written down as[1]

$$\Psi_D = \begin{pmatrix} \psi_L \\ \xi_R \end{pmatrix}, \tag{4}$$

[1] This is the so-called chiral representation. Other unitarily equivalent representations lead to a different choice of γ^μ later on.

with ψ_L and ξ_R being a left- and right handed Weyl spinor respectively[2]. The transformation law is

$$\Psi_D \to \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \Psi_D \tag{5}$$

and under parity inversion

$$\Psi_D \to P\Psi_D = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \Psi_D = \begin{pmatrix} \xi_R \\ \psi_L . \end{pmatrix} . \tag{6}$$

1. Define the quantities

$$\sigma^{\mu\nu} := \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \quad \bar{\sigma}^{\mu\nu} := \frac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \tag{7}$$

and show that the transformation law on Weyl spinors can be written as

$$\psi_L \to \exp(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu})\psi_L, \quad \psi_R \to \exp(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu})\psi_R. \tag{8}$$

Hint: Remember how new generators were defined on the last exercise sheet and express (1) in terms of the original generators. **1 pt.**

2. The so called gamma matrices are given in the chiral representation by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{9}$$

Show that the Lorentz transformation of a Dirac spinor (5) can be written as

$$\Psi_D \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\Psi_D \equiv \Lambda_D \Psi_D, \tag{10}$$

with $S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. **1 pt.**

3. The gamma matrices satisfy a Clifford algebra, the so-called Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \times \mathbf{1}_{4\times 4}. \tag{11}$$

Use this algebra to show that

$$[\gamma^\mu, S^{\rho\sigma}] = m^{[\rho\sigma]\mu}{}_\nu \gamma^\nu, \tag{12}$$

with the generators of the 4-vector representation

$$m^{[\rho\sigma]\mu}{}_\nu = i(\eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu), \tag{13}$$

as introduced on sheet 2. **1 pt.**

[2] Note that a complex 4-vector was also specified by a left- and a right handed Weyl spinor. This is due to the fact that the dimension of the tensor product of two Weyl representations is the same as that of a direct sum. The Lorentz group however acts quite differently.

The commutator relation is equivalent to

$$(1+\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}) = (1-\frac{i}{2}\omega_{\rho\sigma}m^{[\rho\sigma]})^\mu{}_\nu\gamma^\nu. \tag{14}$$

This is the infinitesimal version of $\Lambda_D^{-1}\gamma^\mu\Lambda_D = \Lambda^\mu{}_\nu\gamma^\nu$, which you can use in the next exercise.

4. It should by now be clear that the Lorentz transformation on a field of Dirac spinors $\psi(x)$ is given by
$$\psi(x) \to \Lambda_D\psi(\Lambda^{-1}x). \tag{15}$$
Show that the Dirac equation
$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{16}$$
is invariant under Lorentz transformations. **1 pt.**

5. Show that a Dirac spinor field $\psi(x)$ that satisfies the Dirac equation automatically satisfies the Klein-Gordon equation $(\Box + m^2)\psi(x) = 0$. **1 pt.**

6. Show that the spinor $\bar\psi \equiv \psi^\dagger\gamma^0$ transforms as $\bar\psi \to \bar\psi\Lambda_D^{-1}$ and that the lagrangian
$$\mathcal{L}_{\text{Dirac}} = \bar\psi(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{17}$$
therefore transforms as a Lorentz scalar. What are the equations of motion? **1 pt.**

7. Show that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ and that the equations of motion for $\bar\psi(x)$ can be obtained from that for $\psi(x)$ by taking the hermitian transpose. **1 pt.**

2 Classical solutions (8 pts.)

Since the Dirac field also obeys the Klein-Gordon equation we can directly deduce that a basis of solutions is again given by plane wave solutions of positive and negative frequency, i.e.

$$\psi_+(x) = u(p)e^{-ip\cdot x} \quad \text{and} \quad \psi_-(x) = v(p)e^{ip\cdot x}, \tag{18}$$

with $p^2 = m^2$ and $p^0 > 0$.

1. What constraint is put onto $u(p)$ by the Dirac equation? Show that in the rest frame $p^\mu = (m,0,0,0) = p_0^\mu$ of a massive particle this constraint is solved by
$$u(p_0) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}. \tag{19}$$
1 pt.

We normalise ξ so that $\xi^\dagger\xi = 1$.

2. Apply a boost in z direction to $u(p_0)$ with rapidity η so that $p^\mu = (E,0,0,p^3) = \Lambda^\mu{}_\nu p_0^\nu$ and show that
$$u(p) = \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi \end{pmatrix} \tag{20}$$
Hint: First apply the boost to p_0 and find a relation between E, p^3 and m, η. Then transform $u(p_0)$ using the transformations on left- and right handed Weyl spinors. **2 pt.**

3. Show that this result can be simplified to

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix}. \tag{21}$$

   This expression is valid for arbitrary momenta. **1 pt.**

4. Show that $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 = m^2$. Show furthermore that $u^\dagger(p)u(p) = 2E_p\xi^\dagger\xi$ and $\bar{u}(p)u(p) = 2m\xi^\dagger\xi$, with $\bar{u} = u^\dagger\gamma^0$. **1 pt.**

5. What changes in (21) for the negative frequency solutions? **1 pt.**

6. With an orthogonal basis ξ^s, $s = 1, 2$ satisfying $\xi^\dagger\xi = 1$ you get two linearly independent solutions for $u(p)$,

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ \sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix}. \tag{22}$$

   Show the completeness relation

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = \gamma \cdot p + m. \tag{23}$$

   1 pt.

7. What is the analogon for the negative frequency solutions? **1 pt.**

These relations will be very useful when you calculate actual fermion scattering processes and need to average over initial spins and/or sum over final state spins.