Physikalisches Institut
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Theoretische Physik

Exercise Sheet 6
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SS 15

Exercises Quantum Field Theory I

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http://www.th.physik.uni-bonn.de/klemm/qft1ss15/

–Homework–

1 Canonical quantisation of the electromagnetic field (15 pts.)

The principles we learned quantising the Klein-Gordon and Dirac field also apply to the electromagnetic field, with the additional complication that due to gauge invariance we have a redundancy in our description. In the path integral formulation one can fix the gauge in a way that easily generalises to non-abelian gauge theories which one encounters in the weak and strong interaction. However, in this exercise we will follow the canonical route. As you know, the Lagrangian of pure electrodynamics is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1}$$

1. Introducing a new non-dynamical field κ[1] we can write down a modified Lagrangian

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\kappa}{2}(\partial_\mu A^\mu)^2. \tag{2}$$

   Note that in the Lagrangian formalism the equations of motion for κ enforce the Lorentz gauge condition

$$\partial_\mu A^\mu = 0. \tag{3}$$

   Set $\kappa = 1$, remembering the equations of motion it imposes, and show that up to a total derivative $\mathcal{L}'$ equals

$$\tilde{\mathcal{L}} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \tag{4}$$

   What are the canonical momenta π^μ to the fields A^μ? In the following we will work with $\tilde{\mathcal{L}}$ and later impose the gauge condition as a condition on the states. **1 pt.**

2. Derive the equations of motion for A^μ and argue that you can expand the field as

$$A^\mu(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{\lambda=0}^{3}\left[\epsilon^\mu(p,\lambda)a_{\mathbf{p},\lambda}e^{i\mathbf{p}\cdot\mathbf{x}} + \epsilon^{\mu*}(p,\lambda)a^\dagger_{\mathbf{p},\lambda}e^{-i\mathbf{p}\cdot\mathbf{x}}\right], \tag{5}$$

   with $\omega_p = |\mathbf{p}| = p^0$. Why is the coefficient of the negative frequency part of the form $\epsilon^{\mu*}(p,\lambda)a^\dagger_{\mathbf{p},\lambda}$? Write down the corresponding expansion for the canonical momenta $\pi^\mu(\mathbf{x})$. **1 pt.**

[1] That the field is non-dynamical follows from the absence of time-derivatives of κ in the Lagrangian. This so-called auxiliary field is nothing more than a Lagrange multiplier.

3. Impose the canonical commutation relations

$$[A^\mu(\mathbf{x},t),\pi^\nu(\mathbf{y},t)] = i\eta^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{6}$$
$$[A^\mu(\mathbf{x},t),A^\nu(\mathbf{y},t)] = [\pi^\mu(\mathbf{x},t),\pi^\nu(\mathbf{y},t)] = 0. \tag{7}$$

   Calculate

$$[\partial_\mu A^\mu(\mathbf{x},t),A^\nu(\mathbf{y},t)] \tag{8}$$

   and show that, however tempting it is, imposing the gauge condition $\partial_\mu A^\mu = 0$ directly on the field operator leads to inconsistencies. *Note that the Minkowski metric in the first commutation relation is forced upon us by Lorentz invariance.* **1 pt.**

In a given Lorentz frame we can choose a basis of polarisation vectors starting with two spacelike vectors

$$\epsilon(\mathbf{p},1) = (0,\boldsymbol{\epsilon}(\mathbf{p},1)), \quad \epsilon(\mathbf{p},2) = (0,\boldsymbol{\epsilon}(\mathbf{p},2)), \tag{9}$$

which we demand to be transversal, i.e.

$$p\cdot\epsilon(\mathbf{p},1) = p\cdot\epsilon(\mathbf{p},2) = 0. \tag{10}$$

In addition we take the spacelike longitudinal vector $\epsilon(\mathbf{p},3) = (0,\mathbf{p}/|\mathbf{p}|)$ and the timelike vector $\epsilon(\mathbf{p},0) = (1,\mathbf{0})$. Furthermore we choose $\boldsymbol{\epsilon}(\mathbf{p},1)$ and $\boldsymbol{\epsilon}(\mathbf{p},2)$ so that the basis is real, orthogonal and normalised according to[2]

$$\epsilon^\mu(p,\lambda)\epsilon_\mu(p,\lambda') = \eta_{\lambda\lambda'}. \tag{11}$$

4. Show that

$$a_{\mathbf{p},\lambda} = -i\eta_{\lambda\lambda}\int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\sqrt{2\omega_p}}\epsilon^\mu(p,\lambda)\left[\pi_\mu(\mathbf{x}) + i\omega_p A_\mu(\mathbf{x})\right]. \tag{12}$$

   Derive the corresponding expression for $a^\dagger_{\mathbf{p},\lambda}$ and the commutation relations for the creation and annihilation operators

$$[a_{\mathbf{p},\lambda},a^\dagger_{\mathbf{p}',\lambda'}] = -(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\eta_{\lambda\lambda'},$$
$$[a_{\mathbf{p},\lambda},a_{\mathbf{p}',\lambda'}] = [a^\dagger_{\mathbf{p},\lambda},a^\dagger_{\mathbf{p}',\lambda'}] = 0. \tag{13}$$

   3 pt.

5. Calculate the normal ordered Hamiltonian (that means you bring all the annihilation operators to the right by convention, without picking up singular delta distributions) and show that

$$H = \int\frac{d^3p}{(2\pi)^3}\omega_p\left[-a^\dagger_{\mathbf{p},0}a_{\mathbf{p},0} + \sum_{\lambda=1,2,3}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}\right]. \tag{14}$$

   Use Noether's theorem to calculate the normal ordered momentum operator

$$P = \int\frac{d^3k}{(2\pi)^3}\mathbf{p}\left[-a^\dagger_{\mathbf{p},0}a_{\mathbf{p},0} + \sum_{\lambda=1,2,3}a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}\right]. \tag{15}$$

   3 pt.

[2] Note that in this case $\eta_{\lambda\lambda'}$ should not be seen as a tensor but only encodes the normalisation of the states.

6. Calculate the norm of a one-photon state

$$\langle 1_{\mathbf{p},0}|1_{\mathbf{p},0}\rangle = \langle 0|a_{\mathbf{p},0}a^\dagger_{\mathbf{p},0}|0\rangle. \tag{16}$$

You should notice two things odd, one of them being a delta distribution $\delta^{(3)}(\mathbf{0})$. However, this you already encountered in quantum mechanics and the infinity is due to the same difficulty in normalising infinitely extended plane wave solutions. To remedy this we smear out the momentum using a smooth distribution $f_\lambda(\mathbf{p})$, peaking at the desired momentum and normalised to

$$\int d^3p|f_\lambda(\mathbf{p})|^2 = 1. \tag{17}$$

Calculate the norm of the smeared out state

$$|\tilde{1}_{\mathbf{p},0}\rangle = \int d^3p f_0(\mathbf{p})a^\dagger_{\mathbf{p},0}|0\rangle \tag{18}$$

and show that whats really odd is that this state has a negative norm![3] **1 pt.**

7. Although the occurrence of negative norm states seems fatal it turns out that one can still make the theory work by restricting attention to a subset of states which one deems physical. In our case these are exactly the states which, on average, satisfy the gauge condition:

$$\langle\Phi_{\mathrm{phys}}|\partial_\mu A^\mu(\mathbf{x},t)|\Phi_{\mathrm{phys}}\rangle = 0 \tag{19}$$

We can decompose the field $A^\mu(\mathbf{x},t)$ into positive and negative frequency parts

$$A_\mu(\mathbf{x},t) = A^{(+)}_\mu(\mathbf{x},t) + A^{(-)}_\mu(\mathbf{x},t), \tag{20}$$

consisting only of annihilation or creation operators respectively. Show that the equivalent condition

$$\partial^\mu A^{(+)}_\mu|\Psi_{\mathrm{phys}}\rangle = 0 \tag{21}$$

leads to

$$[a_{\mathbf{p},0} - a_{\mathbf{p},3}]\,|\Psi_{\mathrm{phys}}\rangle = 0 \quad\Leftrightarrow\quad \langle\Psi_{\mathrm{phys}}|\,[a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3}] = 0. \tag{22}$$

1 pt.

8. Show that $a^\dagger_{\mathbf{p},1}|0\rangle$ and $a^\dagger_{\mathbf{p},2}|0\rangle$ are physical states while $a^\dagger_{\mathbf{p},0}|0\rangle$ and $a^\dagger_{\mathbf{p},3}|0\rangle$ are not. Show that $|\phi\rangle = (a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3})|0\rangle$ is physical. **1 pt.**

9. As you know the photon as a massless spin 1 particle has only two physical polarisation states or degrees of freedom. Show that the state $|\phi\rangle$ has vanishing norm, vanishing energy and vanishing momentum. Show that to the energy and momentum expectation value of physical states only the transversal modes contribute. **1 pt.**

[3] Note that this is not due to the specific form of normalisation we choose since a complete basis of polarisations must contain one timelike vector.

In fact its easy to see from (22) that all matrix elements of the form

$$\langle \Psi_{\text{phys}} | \left[(a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3}) | \Psi' \rangle \right] \tag{23}$$

vanish and therefore the states containing „pseudo photons“ fully decouple from the physical sector. We therefore have an equivalence class

$$\Psi_{\text{phys}} \sim \Psi_{\text{phys}} + c|\phi'\rangle \tag{24}$$

where $|\phi'\rangle$ is a physical state containing pseudo photons and can always choose a representative with only transversal excitations.

10. Show that

$$\langle 0 | A_\mu(x) A_\nu(y) | 0 \rangle = -\eta_{\mu\nu} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \exp\left[-ip \cdot (x-y)\right] \tag{25}$$

and with this derive the Feynman propagator

$$D_F^{\mu\nu}(x-y) = \langle 0 | T\left[A^\mu(x) A^\nu(y)\right] | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i\eta^{\mu\nu}}{p^2 + i\epsilon} \exp\left[-ip \cdot (x-y)\right]. \tag{26}$$

Hint: you can use the completeness relation on the polarisation vectors

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon^\mu(\mathbf{p}, \lambda) \epsilon^\nu(\mathbf{p}, \lambda) = \eta^{\mu\nu}. \tag{27}$$

2 pt.

The necessity to sum over all polarisation vectors to obtain the form (26) indicates that although the polarisation modes $\lambda = 0, 3$ are unphysical as initial or final states they have to appear as intermediate particles to make the theory consistent.