Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 7
2.6.2015
SS 15

Exercises Quantum Field Theory I

Prof. Dr. Albrecht Klemm, Thorsten Schimannek

Hand in: 9.6.2015

http://www.th.physik.uni-bonn.de/klemm/qft1ss15/

–Homework–

1 PCT and All That. 10 pts.

As was shown in the lecture parity reversal and charge conjugation are represented by unitary, linear operators P and C, i.e.

$$\mathcal{O}^\dagger = \mathcal{O}^{-1} \quad \text{and} \quad \mathcal{O}\, c = c\, \mathcal{O}\,, \quad \forall c \in \mathbb{C}, \tag{1}$$

for $\mathcal{O} \in \{P,\, C\}$. They act on the quantised Dirac field operators as

$$P\psi(t,\mathbf{x})P = \eta\gamma^0\psi(t,-\mathbf{x}), \quad P\overline{\psi}(t,\mathbf{x})P = \eta^*\overline{\psi}(t,-\mathbf{x})\gamma^0, \tag{2}$$

with $|\eta| = 1$, and

$$C\psi(x)C = -i(\overline{\psi}(x)\gamma^0\gamma^2)^T, \quad C\overline{\psi}(x)C = (-i\gamma^0\gamma^2\psi(x))^T. \tag{3}$$

On the other hand time reversal is represented by a unitary, antilinear operator[1], i.e.

$$T^\dagger = T^{-1} \quad \text{and} \quad T\, c = c^*\, T, \quad \forall c \in \mathbb{C}, \tag{4}$$

which acts like

$$T\psi(t,\mathbf{x})T = \gamma^1\gamma^3\psi(-t,\mathbf{x}), \quad T\overline{\psi}(t,\mathbf{x})T = -\overline{\psi}(-t,\mathbf{x})\gamma^1\gamma^3. \tag{5}$$

1. Show that the Dirac field bilinears and the partial derivatives transform under P, C and T by obtaining the signs listed in the following table, where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$:

	$\overline{\psi}\psi$	$i\overline{\psi}\gamma^5\psi$	$\overline{\psi}\gamma^\mu\psi$	$\overline{\psi}\gamma^\mu\gamma^5\psi$	$\overline{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$\eta^{\mu\mu}$	$-\eta^{\mu\mu}$	$\eta^{\mu\mu}\eta^{\nu\nu}$	$\eta^{\mu\mu}$
T	$+1$	-1	$\eta^{\mu\mu}$	$\eta^{\mu\mu}$	$-\eta^{\mu\mu}\eta^{\nu\nu}$	$-\eta^{\mu\mu}$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

Note: The electromagnetic field A_μ couples to fermions via the term $\mathcal{L}_{int.} = -e\overline{\psi}\gamma^\mu\psi A_\mu$ and from the resulting equations of motions one can conclude that A_μ shares its transformation properties with the bilinear $\overline{\psi}\gamma^\mu\psi$.

Hint: The Dirac matrices are Lorentz scalars and $P\gamma^\mu P = C\gamma^\mu C = \gamma^\mu$ but $T\gamma^\mu T = (\gamma^\mu)^*$.
4 pts.

[1] This is also called antiunitary.

2. Give short answers to the following questions:
   - Why is angular momentum invariant under parity inversion?
   - How does the helicity of a particle transform under parity inversion?
   - How do angular momentum and helicity transform under time inversion?
   - How do the generators of boosts and rotations transform under parity and time inversion respectively?
   - How do the representations change under these actions?
   - Which representations you know could be used to build a parity breaking theory?

   3 pts.

3. Let $\phi(x)$ be a complex-valued Klein-Gordon field. Find unitary operators P, C and an antiunitary operator T that give the following transformations of the Klein-Gordon field:

$$\begin{aligned} P\phi(t,\mathbf{x})P =& \phi(t,-\mathbf{x}) \\ T\phi(t,\mathbf{x})T =& \phi(-t,\mathbf{x}) \\ C\phi(t,\mathbf{x})C =& \phi^*(t,\mathbf{x}) \end{aligned} \tag{6}$$

   It is sufficient to show their action on the creation and annihilation operators. Find the transformation properties of the components of the current

$$J^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \tag{7}$$

   under P, C and T. **2 pts.**

4. Show that any hermitian Lorentz-scalar local operator built from $\psi(x)$, $\phi(x)$, their conjugates and A_μ has $CPT = +1$. **1 pt.**

2 Wick's theorem (5 pts.)

When we evaluate the perturbative expansion of interacting quantum fields it will be necessary to evaluate time ordered products of a large number of field operators. This will be greatly simplified by the use of Wick's theorem. Consider a real scalar field and decompose the operator into positive- and negative frequency parts,

$$\phi_(x) = \phi^+(x) + \phi^-(x), \tag{8}$$

where

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}a_\mathbf{p}e^{-ip\cdot x} \quad \text{and} \quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}a^\dagger_\mathbf{p}e^{ip\cdot x}. \tag{9}$$

The normal ordering operator $N\{\dots\}$ brings all the positive frequency parts in a product of field operators to the right, i.e.

$$N\{\phi_1\dots\phi_n\} = \sum_{\sigma\in\text{perm.}}\sum_{i=1}^{n}\frac{1}{(i-1)!(n-i)!}\phi^-_{\sigma(1)}\phi^-_{\sigma(2)}\cdots\phi^-_{\sigma(i)}\phi^+_{\sigma(i+1)}\cdots\phi^+_{\sigma(n)}, \tag{10}$$

where we defined $\phi_i \equiv \phi(x_i)$. *Notice that the positive- and negative frequency parts commute among themselves.* In particular $\langle 0|N\{\phi_1\dots\phi_n\}|0\rangle = 0$.

1. Show that for the time ordered product of two field operators

$$T\{\phi_1\phi_2\} = N\{\phi_1\phi_2 + \wick{\c\phi_1\c\phi_2}\}, \tag{11}$$

   with $\wick{\c\phi_1\c\phi_2} = D_F(x_1 - x_2)$. **1 pt.**

In general Wick's theorem states that

$$T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)\dots\phi(x_n) + \text{all possible contractions}\}, \tag{12}$$

for example

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} = N\{&\phi_1\phi_2\phi_3\phi_4 + \wick{\c\phi_1\c\phi_2}\phi_3\phi_4 + \wick{\c\phi_1\phi_2\c\phi_3}\phi_4 + \wick{\c\phi_1\phi_2\phi_3\c\phi_4} + \phi_1\wick{\c\phi_2\c\phi_3}\phi_4 \\ &+\phi_1\wick{\c\phi_2\phi_3\c\phi_4} + \phi_1\phi_2\wick{\c\phi_3\c\phi_4} + \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}\}, \end{aligned} \tag{13}$$

with

$$N\{\wick{\c\phi_1\phi_2\c\phi_3}\phi_4\} = D_F(x_1 - x_3)\cdot N\{\phi_2\phi_4\}. \tag{14}$$

2. Prove Wick's theorem by induction on the number of fields. **4 pts.**

Note that Wick's theorem also holds for fermionic fields with

$$\wick{\c\psi(x)\c{\overline{\psi}}(y)} = S_F(x-y), \quad \wick{\c\psi(x)\c\psi(y)} = \wick{\c{\overline{\psi}}(x)\c{\overline{\psi}}(y)} = 0 \tag{15}$$

and the additional rule that the contracted fields have to be brought adjacent to each other, introducing a minus sign for each necessary permutation, e.g.

$$N\{\wick{\c\psi_1\psi_2\c{\overline{\psi}}_3}\overline{\psi}_4\} = -\wick{\c\psi_1\c{\overline{\psi}}_3}N\{\psi_2\overline{\psi}_4\} = -S_F(x_1 - x_3)N\{\psi_2\overline{\psi}_4\}. \tag{16}$$