Physikalisches Institut
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Theoretische Physik

Exercise Sheet 8
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SS 15

Exercises Quantum Field Theory I

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`http://www.th.physik.uni-bonn.de/klemm/qft1ss15/`

–Homework–

1 Wick gymnastics (8 pts)

In the lecture we derived the expression

$$\langle\Omega|T\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_I(x_1)\dots\phi_I(x_n)\exp[-i\int_{-T}^{T}dt\,H_I(t)|\right\}|0\rangle}{\langle 0|T\left\{\exp[-i\int_{-T}^{T}dtH_I(t)|\right\}|0\rangle} \tag{1}$$

for the expectation value of a time ordered product of real, weakly interacting scalar fields with Hamiltonian

$$H = \frac{1}{2}\int d^3x\,(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) + H_I. \tag{2}$$

1. Assume that a nonlinear perturbation is given by

$$H_I = \int d^3x\,\frac{\lambda}{4!}\phi^4 \tag{3}$$

and use Wick's theorem to calculate the numerator of the Greens function,

$$\langle 0|T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)e^{-i\int dt\,H_I}\}|0\rangle, \tag{4}$$

to first order in λ, i.e. neglecting terms of order $\mathcal{O}(\lambda^2)$. Think about the multiplicity of each contraction in order to avoid redundant work. *You do not have to insert explicit expressions for the propagators or carry out the integration.* **3 pts.**

2. Calculate the denominator and show that the contributions of the unconnected vacuum bubbles cancel. **2 pt.**

3. What happens if you calculate the Green's function for an odd number of field operators? **1 pt.**

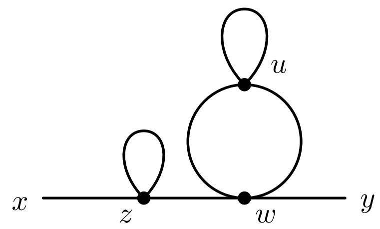
The various contributions can be encoded in the form of *Feynman diagrams*. For this you represent every contraction, i.e. propagator, by a line and every factor of

$$-i\lambda\int d^4x\phi\phi\phi\phi \tag{5}$$

by a vertex. Then you can draw a representative diagram for a certain interaction topology and obtain the multiplicity by combinatorical reasoning. For example the contraction

$$
\begin{aligned}
&\langle 0|\phi(x)\phi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int d^4z\phi\phi\phi\phi \int d^4w\, \phi\phi\phi\phi \int d^4u\, \phi\phi\phi\phi|0\rangle \\
&= \frac{1}{3!}\left(\frac{-i\lambda}{3!}\right)^3 \int d^4z\, d^4w\, d^4u\, D_{xz}D_{zz}D_{zw}D_{wy}D_{wu}^2D_{uu} \qquad (6)
\end{aligned}
$$

can be represented by the following diagram:



The total number of 10,368 contractions corresponding to this diagram can be obtained via

$$
\underbrace{3!}_{\text{interchange of vertices}} \times \underbrace{4\cdot 3}_{\text{placement of contractions into } z \text{ vertex}} \times \underbrace{4\cdot 3\cdot 2}_{\text{placement of contractions into } w \text{ vertex}} \times \underbrace{4\cdot 3}_{\text{placement of contractions into } u \text{ vertex}} \times \underbrace{1/2}_{\text{interchange of } w-u \text{ contractions}} .
$$

However the factor of 3! cancels with the one from the expansion of the exponential and the three four factorials almost cancel with our additional factors of 1/4! in the interaction. The failure in cancellation by a factor of $2\cdot 2\cdot 2$ is due to symmetries of the diagram. Two factors of two come from the lines that start and end on the same vertex, another factor of two comes from the symmetry under interchange of the lines connecting the w and u vertices and other symmetries could be due to the equivalence of two or more vertices.[1]

4. Draw all the diagrams contributing with order λ^2 and write down the corresponding symmetry factors. **2 pts.**

2 A glimpse at the Path integral in zero dimensions (6 pts.)

A nice toy model which captures much of the combinatorics is zero dimensional quantum field theory, also known as ordinary integration. In the path integral formulation of QFT in $d > 0$ correlation functions are given by an infinite dimensional integral

$$
\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = \frac{1}{n}\int[\mathcal{D}\phi]\phi(x_1)\ldots\phi(x_n)\exp(iS), \qquad (7)
$$

with

$$
n = \int[\mathcal{D}\phi]\exp(iS), \qquad (8)
$$

[1] M. E. Peskin, D. V. Schroeder - *An introduction to Quantum Field Theory*, p 92f.

where the integration is over all field configurations, not only those obeying the equations of motion, and $S = \int d^4x\, \mathcal{L}$ is the action. Note that in this formalism $\phi(x)$ is not an operator but an ordinary c-number valued[2] field. In zero dimensions space-time is a point, the field assigns a number to this point and the path integral is just the ordinary integration over the possible numbers. In the free theory, due to the lack of directions, only the mass term makes sense. Remember the gaussian integral

$$Z_0 = \int_{-\infty}^{\infty} dx\, e^{-\frac{m^2}{2}x^2} = \frac{\sqrt{2\pi}}{m}. \tag{9}$$

1. Introduce the linear coupling to an external field and calculate

$$Z(J) = \int_{-\infty}^{\infty} dx\, e^{-\frac{m^2}{2}x^2 + Jx}. \tag{10}$$

   Note that you can use derivatives $\frac{\partial^n}{\partial J^n} Z(J)\big|_{J=0}$ to obtain the expectation values of powers of x. Write down the general expression for $\frac{\partial^n}{\partial J^n} Z(J)\big|_{J=0}$. **1 pt.**

2. Introduce a nonlinear perturbation

$$Z_0' = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2 + \frac{\lambda}{4!}x^4} \tag{11}$$

   and calculate the expectation value

$$\langle\Omega|x^4|\Omega\rangle = \frac{1}{Z_0'} \int_{-\infty}^{\infty} dx\, x^4 e^{-\frac{1}{2}x^2 + \frac{\lambda}{4!}x^4} \tag{12}$$

   up to order one, i.e. neglecting terms of order $\mathcal{O}(\lambda^2)$. Compare the multiplicities with the ones obtained for the real scalar field in four dimensions. **2 pts.**

3. The contributing terms can also be encoded in diagrams involving propagators and vertices. Develop the set of rules and use these to give the contribution at order λ^2. **3 pts.**

In the zero dimensional case corresponding, for a general number of fields, to finite dimensional integrals, we are basically calculating moments of probability distributions and in this case Wick's theorem is known as Isserlis' theorem.

[2]Fermionic fields are actually valued in Grassman numbers which anticommute.