Physikalisches Institut
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Theoretische Physik

Exercise Sheet 9
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SS 15

Exercises Quantum Field Theory I

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http://www.th.physik.uni-bonn.de/klemm/qft1ss15/

–Homework–

1 The S-Matrix (15 pts.)

To cut some corners on the way to real-world calculations we postulate the necessary formulae which will be proven at a later point in the lecture. In order to obtain measurable quantities we first have to calculate transition amplitudes

$$\lim_{T\to\infty} \langle \mathbf{p}_1\mathbf{p}_2\dots|\exp^{-iH(2T)}|\mathbf{k_1}\mathbf{k_2}\dots\rangle \equiv \langle \mathbf{p}_1\mathbf{p}_2\dots|S|\mathbf{k_1}\mathbf{k_2}\dots\rangle, \tag{1}$$

between n particles with momenta $\mathbf{k}_1, \mathbf{k}_2, \dots$ in the infinite past and m particles with momenta $\mathbf{p}_1, \mathbf{p}_2, \dots$ in the infinite future. We will restrict ourselves to the case of two incoming particles. Note that for a general species of particles one also has to specify the discrete quantum numbers in the states. The so-called *S-matrix* is just the time-evolution operator in the limit of very large T. It is convenient to decompose it as

$$S = 1 + iT, \tag{2}$$

so that the *T-matrix* encodes only the non-trivial interactions. We introduce the *invariant matrix element* $\mathcal{M}(k_1, k_1 \to p_1, p_2, \dots)$ via

$$\langle \mathbf{p}_1\mathbf{p}_2\dots|iT|\mathbf{k}_1\mathbf{k}_2\rangle = (2\pi)^4\delta^{(4)}\left(k_1 + k_2 - \sum_i p_i\right)\cdot i\mathcal{M}(k_1, k_1 \to p_1, p_2, \dots) \tag{3}$$

to get rid of an overall momentum conserving delta distribution which is common to all scattering processes. However in formula (1) and (3), $|\mathbf{k_1}\mathbf{k_2}\dots\rangle$ and $|\mathbf{p_1}\mathbf{p_2}\dots\rangle$ are eigenstates of the full Hamiltonian $H = H_0 + H_{int}$ at some common reference time. As we needed to express the interacting vacuum in terms of the free vacuum and interaction picture fields, we need a similar expression relating the multi-particle states to the eigenstates

$$|\mathbf{k_1}\mathbf{k_2}\dots\rangle_0 = \prod_i \sqrt{E_{\mathbf{k}_i}}a^\dagger_{\mathbf{k}_i}|0\rangle \tag{4}$$

of the unperturbed theory[1]. It is obvious that the technique we used for $|\Omega\rangle$ cannot be adapted in a simple way. Without derivation we give the result

$$i\mathcal{M}(k_1, k_1 \to p_1, p_2, \dots)\cdot(2\pi)^4\delta^{(4)}\left(k_1 + k_2 - \sum_i p_i\right) = \left(\begin{array}{c}\text{sum of all connected, amputated Feynman}\\ \text{diagrams with } k_1, k_2 \text{ incoming and } p_1, p_2, \dots \text{ outgoing}\end{array}\right). \tag{5}$$

The prescription on the right hand side certainly needs some explanation.

[1] Here the creation operators $a^\dagger_{\mathbf{k}_i}$ are the same that occur in the expansion of the intraction picture field $\phi_I(x)$

1. The required connectedness is stricter than the absence of vacuum bubbles. We demand that all the external particles have to participate in the process. An equivalent requirement is that you can not draw a line seperating parts of the diagram without severing a propagator.

2. Amputated means that you cut off the bubbles from the particles propagating to the interaction:

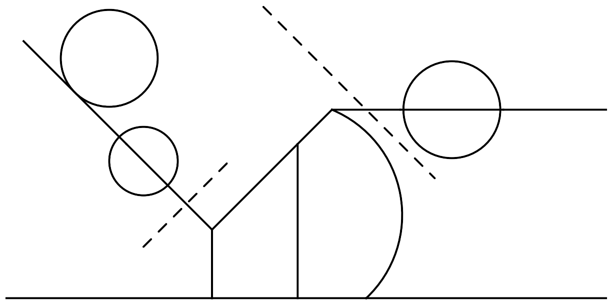
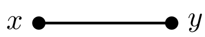
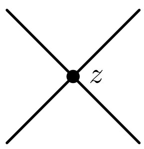
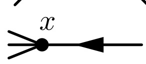


Figure 1: The bubbles at the external propagators have to be amputated along the dashed lines.

The (position space) Feynman rules to calculate the contribution of a diagram to a transition amplitude in ϕ^4 theory are

1. For each propagator,  $= D_F(x-y)$;

2. For each vertex,  $= (-i\lambda)\int d^4z$;

3. For each external line,  $= e^{-ip\cdot x}$;

4. Divide by the symmetry factor.

Your exercise is to translate these rules into momentum space, find corresponding rules for the Dirac- and the electromagnetic field and calculate an elementary process in quantum electrodynamics.

1. Show that the integration introduced by each vertex "absorbs" external lines or parts of the exponential in the integrand of each propagater it is connected to and turns them into a momentum conserving delta distribution. Why will there always be an overall momentum conserving delta distribution associated with each diagram? For what kind of diagrams are all internal momenta fixed by momentum conservation? **2 pt.**

2. Write down the momentum space Feynman rules, so that

$$i\mathcal{M} = \text{sum of all connected, amputated diagrams} \tag{6}$$

and there is no reference to space-time points left. **2 pts.**

3. The position space Feynman rule for the external leg follows from "contraction" of an interaction field with an external particle

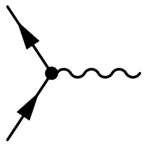
$$\begin{aligned} \phi_I^+(x)|\mathbf{k}\rangle_0 =& \phi_I^+(x)\sqrt{E_\mathbf{k}}a_\mathbf{k}^\dagger|0\rangle = e^{-ik\cdot x}|0\rangle, \\ {}_0\langle\mathbf{k}|\phi_I^- =& \langle 0|e^{ik\cdot x}. \end{aligned} \tag{7}$$

   Use your knowledge about the quantised Dirac- and electromagnetic field to calculate how the rule changes for fermions, antifermions and photons. Give the momentum space rule for incoming, as well as for outgoing particles. **3 pts.**

4. For bosons the direction of momentum along internal lines is arbitrary. For fermions this is different. How is the momentum flow related to the particle number flow? **2 pts.**

5. Guess the momentum space Feynman rules for propagators of the Dirac- and electromagnetic field. For fermions let the arrow on a propagator indicate the direction of particle number flow. **1 pt.**

Now consider a theory of one species of fermions coupled to an electromagnetic field via $H_{int} = -ie\bar{\psi}\gamma^\mu\psi A_\mu$. In addition to the rules for the fermions and the electromagnetic field there is now a vertex with momentum space Feynman rule



$$\mu \quad = -ie\gamma^\mu.$$

6. Show that the leading-order contribution to the scattering of two distinguishable fermions with ingoing momenta k, k' and outgoing momenta p, p' is given by

$$i\mathcal{M} = (-ie)^2\bar{u}(p')\gamma^\mu u(p)\frac{-ig_{\mu\nu}}{(p'-p)^2}\bar{u}(k')\gamma^\nu u(k). \tag{8}$$

   Hint: Think about which diagrams are excluded by the requirement of distinguishability. **1 pt.**

7. Take the non relativistic limit and use the Born approximation

$$\langle p'|iT|p\rangle = -i\tilde{V}(\mathbf{p'} - \mathbf{p})(2\pi)\delta(E_{\mathbf{p'}} - E_\mathbf{p}), \tag{9}$$

   to obtain the momentum space potential $\tilde{V}(q)$ for one of the scattered particles. Calculate the Fourier transform to obtain the Coloumb potential. Is it attractive or repulsive? **3 pts.**

8. What changes for the scattering of a fermion and an antifermion? **1 pt.**