Physikalisches Institut
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Theoretische Physik

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Exercise 1

Exercises Advanced Theoretical Physics

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1 The Mott formula

In this exercise we want to repeat some important concepts of last term by calculating the relativistic cross section for Coulomb scattering at tree level without quantizing the electric field $A_\mu(x)$. The interaction term reads

$$H_I = \int d^3x e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{1.1}$$

1. Show that the T-matrix element for a scattered electron by the electromagnetic field with incoming four-momentum p_i and outgoing four-momentum p_f is to lowest order in α' given by
$$\langle p_f|iT|p_i\rangle = -ie\bar{u}(p_f)\gamma^\mu u(p_i)\tilde{A}_\mu(p_f - p_i), \tag{1.2}$$
where $\tilde{A}_\mu$ denotes the Fouriertransform of A_μ.

2. Show that one can rewrite this for a time-independent potential as
$$\langle p_f|iT|p_i\rangle = i\mathcal{M}(2\pi)\delta(E_f - E_i) \tag{1.3}$$
and calculate $\tilde{A}_\mu$ for the case of a Coulomb potential $A^0 = Ze/4\pi r$.

3. The differential cross section for a process $p_i \longrightarrow p_f$ is given by
$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}|^2. \tag{1.4}$$
Show that for our process the differential cross section is given by
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4\mathbf{p}^2\beta^2\sin^4(\theta/2)}\big(1 - \beta^2\sin^2(\theta/2)\big). \tag{1.5}$$

2 Feynman parameters I

This exercise is devoted to prove some important formulae that are needed in order to calculate loop amplitudes.

1. Show that
$$\frac{1}{AB} = \int_0^1 \frac{1}{\big(xA + (1-x)B\big)^2}dx = \int_0^1 dx \int_0^1 dy\delta(x+y-1)\frac{1}{\big(xA + yB\big)^2}. \tag{2.1}$$

2. Show that
$$\frac{1}{AB^n} = \int_0^1 dx \int_0^1 dy\delta(x+y-1)\frac{ny^{n-1}}{\big(xA + yB\big)^{n+1}}. \tag{2.2}$$

Homework

3 The Rosenbluth formula **(8 points)**

As discussed in the lecture, the interaction vertex of a Dirac fermion can be written quite generally in terms of two form factors $F_1(q^2)$ and $F_2(q^2)$. In this exercise you are asked to deduce the cross section for the scattering of an electron from a proton that is initially at rest. In this particular problem F_1 and F_2 describe the structure resulting from the strong interaction that can (so far) only be determined by measurement. We use the following notation: p_f and p_i denote the four-momentum of the outgoing and ingoing proton (having mass M) respectively whereas k_f and k_i denote the momenta of the electron (having mass $m \ll E$). Furthermore we introduce

$$q = k_i - k_f = p_f - p_i \tag{3.1}$$

which describes the momentum of the virtual photon the structure constants depend on.

1. Explain in a few words why the most general interaction vertex is given by

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2). \tag{3.2}$$

2. Draw the Feynman diagram for this process and calculate the amplitude for this process. Using the Gordon identity

$$\bar{u}(p_f)\gamma^\mu u(p_i) = \bar{u}(p)\big(\frac{p_i^\mu + p_f^\mu}{2M} + i\frac{\sigma^{\mu\nu} q_\nu}{2M}\big)u(p_i), \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \tag{3.3}$$

you should find that

$$i\mathcal{M} = i\frac{e^2}{q^2}\bar{u}(k_f)\gamma_\mu u(k_i)\bar{u}(p_f)\big(\gamma^\mu(F_1+F_2) - \frac{p_f^\mu + p_i^\mu}{2M}F_2\big)u(p_i). \tag{3.4}$$

3. Evaluate $|\mathcal{M}|^2$ and average over spins. You should finally end up with

$$\begin{aligned} |\mathcal{M}|^2 \quad = \quad & \frac{8e^4}{q^4}\Big(\big(F_1+F_2\big)^2\big((k_f\cdot p_f)(k_i\cdot p_i) + (k_f\cdot p_i)(k_i\cdot p_f) - M^2(k_f\cdot k_i) \\ & -m^2(p_f\cdot p_i) + 2m^2M^2\big) + \big(\frac{p_f\cdot p_i + M^2}{4M^2}F_2^2 - F_2(F_1+F_2)\big) \\ & \cdot\big(k_f\cdot(p_f+p_i)k_i\cdot(p_f+p_i) - \frac{1}{2}(k_f\cdot k_i - m^2)(p_f+p_i)^2\big)\Big). \end{aligned} \tag{3.5}$$

Hint: If you do not know them by heart by now, you might find the following identities useful

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \quad \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{3.6}$$

You are not asked to prove them!

4. Now we turn to the kinematics. In the frame where the proton is initially at rest we have

$$p_i = (M, \vec{0}), \quad k_i = (E_i, E_i \vec{e}_z), \quad k_f = (E_f, 0, E_f \sin(\theta), E_f \cos(\theta)). \tag{3.7}$$

Here we used the high energy approximation and denoted the scattering angle by θ. Show that

a)

$$q^2 = -\frac{4E_i^2 \sin^2(\theta/2)}{1 + \frac{2E_i}{M} \sin^2(\theta/2)} \tag{3.8}$$

b)

$$\Big((k_f \cdot p_f)(k_i \cdot p_i) + (k_f \cdot p_i)(k_i \cdot p_f) - M^2(k_f \cdot k_i) - m^2(p_f \cdot p_i) + 2m^2M^2\Big) = \frac{q^4}{4} \tag{3.9}$$

c)

$$\Big(\frac{p_f \cdot p_i + M^2}{4M^2} F_2^2 - F_2(F_1 + F_2)\Big) = \frac{1}{2}\Big((F_1 + F_2)^2 - \big(F_1^2 - \frac{q^2}{4M^2} F_2^2\big)\Big) \tag{3.10}$$

d)

$$\big(k_f \cdot (p_f + p_i) k_i \cdot (p_f + p_i) - \frac{1}{2}(k_f \cdot k_i - m^2)(p_f + p_i)^2\big) = 4(E_i M)^2 + 2E_i M q^2 + \frac{q^4}{4} + q^2 M^2 - \frac{q^4}{4} \tag{3.11}$$

Hint: It might be useful to work out first trivial relations like $(p_f \cdot p_i) = M^2$ - $\frac{q^2}{2}$ and to use (3.1). Furthermore only use the explicit, θ-dependent form of k_f, p_f, if it makes sense!

5. Finally give the full cross section, where you may use that

$$\frac{d\sigma}{d\cos\theta} = \frac{|\bar{\mathcal{M}}|^2}{32\pi M^2 \big(1 + \frac{2E_i}{M} \sin^2(\theta/2)\big)^2}. \tag{3.12}$$

As the final result you should find

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 \Big(\big(F_1^2 - \frac{q^2}{4M^2} F_2^2\big) \cos^2(\theta/2) - \frac{q^2}{2M^2} \big(F_1 + F_2\big)^2 \sin^2(\theta/2)\Big)}{2E_i^2 \big(1 + \frac{2E_i}{M} \sin^2(\theta/2)\big) \sin^4(\theta/2)}. \tag{3.13}$$

4 Feynman parameters II **(2 points)**

1. Show that

$$\frac{1}{A_1 \cdot \ldots \cdot A_n} = \int_0^1 dx_1 \ldots \int_0^1 dx_n \delta(\sum x_i - 1) \frac{(n-1)!}{\big(x_1 A_1 + \ldots + x_n A_n\big)^n}. \tag{4.1}$$

2. Show that

$$\frac{1}{A_1^{m_1} \cdot \ldots \cdot A_n^{m_n}} = \int_0^1 dx_1 \ldots \int_0^1 dx_n \delta(\sum x_i - 1) \frac{\prod x_i^{m_i - 1}}{\big(\sum x_i A_i\big)^{\sum m_i}} \frac{\Gamma(\sum m_i)}{\Gamma(m_1) \cdot \ldots \cdot \Gamma(m_n)}. \tag{4.2}$$