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Exercises Quantum Field Theory II

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`http://www.th.physik.uni-bonn.de/klemm/qft2ws1516/`

–Homework–

1 Chiral anomalies Ìa Fujikawa (12 pts.)

Consider the standard QCD lagrangian for a massless quark

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + i\bar{q}\not{D}q, \tag{1}$$

with $\not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + igA_\mu)$, $A_\mu = A^a_\mu T^a$ and $F_{\mu\nu} = F^a_{\mu\nu}T^a$, where

$$F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - gf^{abc}A^b_\mu A^c_\nu, \tag{2}$$

and T^a being the generators of $SU(3)$, for which the Lie-bracket reads $[T^a, T^b] = if^{abc}T^c$.

1. Show that a chiral transformation $q \to \exp\left[i\alpha\gamma^5/2\right] q$ leaves the lagrangian unchanged. **1pt.**

2. Now let us consider the more general case in wich the chiral transformation is local, i.e. $\alpha = \alpha(x^\mu)$. Show that in this case

$$\mathcal{L} \to \mathcal{L} + \alpha(x^\mu)\partial_\mu j^\mu_5, \quad \text{with} \quad j^\mu_5 = \bar{q}\gamma^\mu\gamma_5 q. \tag{3}$$

   1 pt.

Now consider the path integral

$$Z = \int \mathcal{D}A^a_\mu \mathcal{D}q\mathcal{D}\bar{q}\exp\left[i\int d^4x\mathcal{L}\right]. \tag{4}$$

3. Argue that the path integral measure $\mathcal{D}q\mathcal{D}\bar{q}$ is not invariant under chiral transformations. **1 pt.**

To study this in more detail we expand q and $\bar{q}$ in the basis of eigenstates of the Dirac operator

$$q = \sum_n a_n\phi_n(x^\mu), \quad \bar{q} = \sum_n \hat{a}_n\hat{\phi}_n(x^\mu), \tag{5}$$

with

$$(i\not{D})\phi_m = \lambda_m\phi_m, \quad \hat{\phi}_m(i\not{D}) = \lambda_m\hat{\phi}_m \tag{6}$$

and a_n, $\hat{a}_n$ being Grassmann coefficients. With this we can write the path integral measure as

$$\mathcal{D}q\mathcal{D}\bar{q} = \prod_m da_m d\hat{a}_m. \tag{7}$$

4. Show that chiral transformations act on the Grassmann coefficients according to

$$a_m \to \sum_n \left(\delta_{n,m} + \frac{1}{2} \underbrace{\int d^4 x i\alpha(x) \hat{\phi}_m \gamma^5 \phi_n}\, C_{mn} + \dots \right) a_n \tag{8}$$

Hint: Use the orthogonality relations $\int d^4 x \hat{\phi}_n \phi_m = \delta_{n,m}$. **1 pt.**

5. Use the previous result together with (7) to show that for an infinitesimal chiral transformation one has

$$\mathcal{D}q\mathcal{D}\bar{q} \to \exp\left[-\mathrm{tr}\, C\right] \mathcal{D}q\mathcal{D}\bar{q}. \tag{9}$$

Hint: det $A = e^{tr(\log A)}$. **1 pt.**

Let us examine our result in more detail. The trace term containes a γ_5 which might suggest that it vanishes, however we are taking an infinite sum. To properly study its behavior we introduce a regulator

$$\sum_n \bar{\phi}_n \gamma^5 \phi_n = \lim_{M\to\infty} \sum_n \bar{\phi}_n \gamma^5 \phi_n e^{\lambda_n^2/M^2}. \tag{10}$$

6. Prove that

$$\lambda_m^2 = \left(i\not{D}\right)^2 = -D^2 + \frac{g}{2}\sigma^{\mu\nu} F_{\mu\nu}, \tag{11}$$

with $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$. **2 pts.**

Since we are taking the limit $M \to \infty$ only small wavelengths contribute. This justifies an expansion of the exponential in terms of $\sigma \cdot F$. Tracing over Dirac indices and ignoring the background A_μ we obtain

$$\sum_n \bar{\phi}_n \gamma^5 \phi_n = \lim_{M\to\infty} \mathrm{tr}\left[\gamma^5 \frac{1}{2}\left(\frac{g}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right] \langle x| e^{-\frac{\partial}{M^2}} |x\rangle \tag{12}$$

at leading order.

7. Why does the linear term in $\sigma \cdot F$ vanish? **.5 pts.**

8. Evaluate the expression $\langle x| e^{-\frac{\partial^2}{M^2}} |x\rangle$ as an integral in momentum space. **2 pts.**

9. Take the trace to finally show that a chiral transformation in the path integral can be reabsorbed in the exponential i.e.

$$Z \to \int \mathcal{D}A^a_\mu \mathcal{D}q\mathcal{D}\bar{q} \exp\left[i \int d^4x \left\{\mathcal{L} + \alpha(x^\mu)\left(\partial_\mu j_5^\mu + \frac{g^2}{32\pi^2}\mathrm{tr}\left[F_{\mu\nu}\tilde{F}^{\mu\nu}\right]\right)\right\}\right], \tag{13}$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

This functional derivation is due to Fujikawa and leads straightforwardly to the result

$$\partial_\mu j_5^\mu = -\frac{g^2}{32\pi^2}\mathrm{tr}\left[F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{14}$$