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Exercises Quantum Field Theory II

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`http://www.th.physik.uni-bonn.de/klemm/qft2ws1516/`

–Homework–

1 Vacuum polarization (25 pts.)

Loop corrections due to intermediate fermion/anti-fermion pairs in the propagation of a gauge boson effectively turn the vacuum into a dielectric medium and make the coupling constant a function of the gauge boson momentum. This effect is called vacuum polarization. In this exercise you will calculate and interprete the one-loop correction to the electric charge in QED.

1. The diagram responsible for the order α correction to the electric charge is shown in figure 1. Argue with lorentz symmetry and the QED Ward identity that the contribution of this

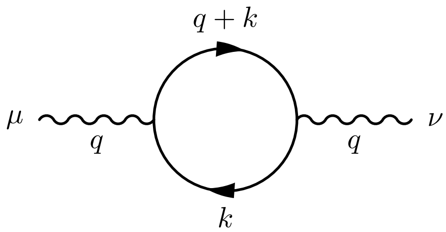

Figure 1: One loop contribution to vacuum polarization in QED.

   diagram is of the form

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2). \tag{1}$$

   2 pts.

2. Argue that the full contribution is obtained by summing chains of 1-PI diagrams as indicated in figure 2. **1 pt.**

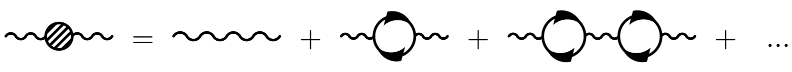

Figure 2: The full contribution is obtained by summing chains of 1-PI diagrams.

3. Show that this leads to the general form

μ  ν $= \frac{-i\eta_{\mu\nu}}{q^2(1-\Pi(q^2))}$,

where you can use that terms proportional to q_μ or q_ν won't contribute to the S-matrix, again due to the Ward identity. **2 pts.**

For small momentum we can define

$$\frac{1}{1-\Pi(0)} \equiv Z_3. \tag{2}$$

This can be absorbed in the factors of e coming from the vertices to which this diagram couples. We therefore have a *physical charge* e which is related via $e = \sqrt{Z_3}e_0$ with the *bare charge* e_0 occuring as a coupling constant in the Lagrangian.

4. Show that

$$\frac{-i\eta^{\mu\nu}}{q^2}\left(\frac{e_0^2}{1-\Pi(q^2)}\right) = \frac{-i\eta^{\mu\nu}}{q^2}\left(\frac{e^2}{1-[\Pi_2(q^2)-\Pi_2(0)]}\right) + \mathcal{O}(\alpha^2), \tag{3}$$

where $\Pi(q^2) = \Pi_2(q^2) + \mathcal{O}(\alpha^2)$. **2 pts.**

5. Calculate $i\Pi^{\mu\nu}(q)$ using the Feynman rules for invariant amplitudes and evaluate the trace. Why do you get an overall minus sign for the fermion loop? **4 pts.**

6. Combine the two factors in the denominator introducing a Feynman parameter and shift the integration variable to obtain

$$i\Pi_2^{\mu\nu}(q) = -4e^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \frac{2l^\mu l^\nu - \eta^{\mu\nu}l^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}(m^2 + x(1-x)q^2)}{(l^2-\Delta+i\epsilon)^2}, \tag{4}$$

with $l = k + xq$ and $\Delta = m^2 - x(1-x)q^2$. **3 pts.**

7. Wick rotate the integration contour avoiding the poles and go to $d = 4-\epsilon$ dimensions to regularize the integral. Note that you can replace $l^\mu l^\nu \to \frac{1}{d}l^2\eta^{\mu\nu}$ by symmetry. The electric charge is dimensionless and does not need to be rescaled as it was necessary with the coupling constant in ϕ^4 theory. **4 pts.**

8. Use the formulae

$$\frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}, \tag{5}$$

$$\frac{d^d l_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}, \tag{6}$$

to calculate

$$\Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\log\left(\frac{m^2}{m^2-x(1-x)q^2}\right). \tag{7}$$

5 pts.

9. Show that in the limit of deep elastic scattering, i.e. $-q^2 \gg m^2$,

$$\alpha(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \log\left(\frac{-q^2}{Am^2}\right)}, \tag{8}$$

with $A = \exp(5/3)$. **2 pts.**