Physikalisches Institut
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Theoretische Physik

Exercise sheet 3
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Exercises Quantum Field Theory II

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http://www.th.physik.uni-bonn.de/klemm/qft2ws1516/

–HOMEWORK–

1 The path integral in quantum mechanics (10 pts.)

In this exercise you will recall some basics about the path integral in ordinary quantum mechanics, i.e. 0+1 dimensional quantum field theory. Consider a quantum mechanical system with a time independent Hamiltonian H The propagator $G(t, x_f, x_i)$ is defined by

$$G(t, x_f, x_i) = \langle x_f| \exp\left[-itH\right] |x_i\rangle. \tag{1}$$

1. Show that the propagator obeys

$$G(t, x_f, x_i) = \int dx\, G(t-t', x_f, x)G(t', x, x_i), \quad \text{for} \quad 0 < t' < t \tag{2}$$

   and that $G_r(t, x_f, x_i) = \Theta(t)G(t, x_f, x_i)$ is a retarded Greens-function of the Schrödinger equation, i.e.

$$\left[i\partial_t - H(x_f, -i\partial_{x_f})\right] G_r(t, x_f, x_i) = i\delta(t)\delta(x_f - x_i). \tag{3}$$

   3 pts.

2. Split the interval t into N equal parts $t = \epsilon N$ to show

$$G(t, x_N, x_0) = \int \left(\prod_{k=1}^{N-1} dx_k\right) \prod_{k=1}^{N} \langle x_k| \exp\left(-i\epsilon H\right) |x_{k-1}\rangle \tag{4}$$

   2 pt.

3. Split the Hamiltonian $H = \frac{p^2}{2m} + V(x) = T + V$ into kinetic and potential parts as usual.[1] Use the Baker-Campbell-Hausdorff formula to argue that in the limit $N \to \infty$ with $N\epsilon = t = const.$ one may write

$$\begin{aligned} G(\epsilon, x_k, x_{k-1}) &\approx \langle x_k| \exp\left(-i\epsilon T\right) |x_{k-1}\rangle \exp\left[-i\epsilon V(x_{k-1})\right] \\ &= \sqrt{\frac{m}{2\pi i\epsilon}} \exp\left\{i\epsilon\left[\frac{m}{2}\frac{(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1})\right]\right\}, \end{aligned} \tag{5}$$

   Hint: The overlap of momentum- and position eigenstates is $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx}$. **3 pts.**

[1] For convenience we assume that the potential only depends on the position, but this is not essential. In fact the path integral will allow you to treat momentum dependent couplings in quantum field theory in a much more straightforward manner.

4. Argue that in the limit $N \to \infty, N\epsilon = t$ one can write

$$G(t, x_f, x_i) = \int \mathcal{D}[x] e^{iS[x]}, \tag{6}$$

with

$$\mathcal{D}[x] = \lim_{\substack{N \to \infty \\ N\epsilon = t}} \int \left(\prod_{k-1}^{N-1} dx_k \sqrt{\frac{m}{2\pi i \epsilon}} \right) \quad \text{and} \quad S[x] = \int_0^t ds \left[\frac{1}{2} m \dot{x}^2 - V(x) \right].$$

What is the interpretation of this formula? **2 pts.**

2 Multi-dimensional gaussian integration (10 pts.)

As was the case with canonical quantisation, to make sense out of the path integral for an interacting theory we have to solve the non-interacting theory first. You will see in the lecture how this task involves multi-dimensional gaussian integrals of the form[2]

$$\int dx^d e^{-\frac{1}{2}\vec{x}^t A \vec{x}} \tag{7}$$

where A is a symmetric $d \times d$ matrix. You will solve this integral and some relatives which will be essential for dealing with QFTs perturbatively.

1. Show that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} = \sqrt{2\pi}. \tag{8}$$

Hint: Calculate the square of the integral and go to polar coordinates. **2 pts.**

2. Use this result to show

$$\int dx^d e^{-\frac{1}{2}\vec{x}^t A \vec{x}} = \sqrt{\frac{(2\pi)^N}{\text{Det}\,[A]}} \tag{9}$$

Hint: Use the fact that every symmetric matrix can be diagonalized by an orthogonal basis transformation. **2 pts.**

3. Now derive

$$\int dx^d e^{-\frac{1}{2}\vec{x}^t A \vec{x} + \vec{J}\cdot\vec{x}} = \sqrt{\frac{(2\pi)^N}{\text{Det}\,[A]}} e^{\frac{1}{2}\vec{J}^t A^{-1} \vec{J}} \tag{10}$$

2 pts.

4. Define

$$\langle x_i \dots x_j \rangle = \frac{\int dx^d x_i \dots x_j e^{-\frac{1}{2}\vec{x}^t A \vec{x}}}{\int dx^d e^{-\frac{1}{2}\vec{x}^t A \vec{x}}} \tag{11}$$

and use the last result to show $\langle x_i x_j \rangle = A^{-1}_{ij}$. **2 pts.**

5. Show how expectation values of arbitrary $x_i's$ can be calculated by “Wick contractions”. **2 pts.**

[2] For a path integral the matrix A will be replaced by an operator. However, the notions of eigenvalues and determinants and in particular much of the finite dimensional intuition still make sense.