Physikalisches Institut
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Theoretische Physik

Exercise sheet 7
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WS 15/16

Exercises Quantum Field Theory II

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http://www.th.physik.uni-bonn.de/klemm/qft2ws1516/

–Homework–

1 You can't say functional without fun (15 pts.)

In the lecture you introduced couplings to a non-dynamical field $J(x)$ to derive the generating functional for the free scalar field

$$Z[J] = Z_0 \cdot \exp\left[-\frac{i}{2}\int\int d^4x d^4y J(x) D_F(x-y) J(y)\right] = Z_0 \cdot e^{iW[J]}. \tag{1}$$

These coupling are not only useful to do perturbative expansions but also describe the interaction with a background field/current.

1. Consider a background field with two infinitely huge lumps

$$J(x) = \delta^{(3)}(\vec{x}-\vec{x}_1) + \delta^{(3)}(\vec{x}-\vec{x}_2) \tag{2}$$

   and why not just call them particles. Forget about the self-interactions (i.e. terms with $J_1 D_F J_1$ or $J_2 D_F J_2$) and derive the contribution of the remaining terms to $W[J]$,

$$W_{12}[J] = T\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\cdot\vec{k}(\vec{x}_1-\vec{x}_2)}}{k^2+m^2}, \tag{3}$$

   where T is the time between our initial and final boundary conditions. **3 pts.**

2. Argue that $iW[J] = -iE[J]T$, where $E[J]$ is the energy of the self-energy of the background field. Use

$$\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\cdot\vec{k}(\vec{x}_1-\vec{x}_2)}}{k^2+m^2} = \frac{1}{4\pi r}e^{-mr} \tag{4}$$

   to conclude that exchange of scalar particles leads to a Yukawa-potential. In particular the force is always attractive. **2 pts.**

We want to do the same calculation for the exchange of spin 1 and spin 2 particles. To avoid the additional complications due to local gauge invariance we assume that the vector- and tensor-bosons have a small mass.

3. A massive vector boson has three degrees of polarization. In the rest frame $k^\mu = (m, 0, 0, 0)$ we choose the basis

$$\epsilon_\mu^{(a)} = \delta_\mu^a, \quad a \in \{1, 2, 3\} \tag{5}$$

from which the general basis $\epsilon_\mu^{(a)}(k)$ is obtained via boosts. The propagator is then of the form[1]

$$D_{\mu\nu} = \frac{\sum\limits_a \epsilon_\mu^{(a)}(k)\epsilon_\nu^{(a)}(k)}{k^2 - m^2}. \tag{6}$$

Show that

$$\sum_a \epsilon_\mu^{(a)}(k)\epsilon_\nu^{(a)}(k) = -\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right). \tag{7}$$

2 pts.

4. The Lagrangian of a massive vector boson coupled to an external field is given by

$$\mathcal{L} = \frac{1}{2} A_\mu \left[(\partial^2 + m^2)\eta^{\mu\nu} - \partial^\mu \partial^\nu\right] A_\nu + A_\mu J^\mu, \tag{8}$$

where $J^\mu(x)$ is a background current. Note that in particular $J^0(x)$ is the background charge. Derive from the generating functional of the path integral that opposite charges attract each other while like charges repel. **2 pts.**

5. Lets go on to tackle massive gravity. A massive spin 2 particle has 5 degrees of polarisation for which we choose a basis $\epsilon_{\mu\nu}^{(a)}$. The elements $\epsilon_{\mu\nu}^{(a)}$ are symmetric in the indices μ and ν, satisfy $k^\mu \epsilon_{\mu\nu}^{(a)} = 0$ and are traceless, i.e. $\eta^{\mu\nu}\epsilon_{\mu\nu}^{(a)} = 0$. We fix the normalisation choosing

$$\sum_a \epsilon_{12}^{(a)}(k)\epsilon_{12}^{(a)}(k) = 1. \tag{9}$$

Show that

$$\sum_a \epsilon_{\mu\nu}^{(a)}(k)\epsilon_{\lambda\sigma}^{(a)}(k) = (G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}, \tag{10}$$

with $G_{\mu\nu}(k) = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$. How does the propagator look like? **4 pts.**

6. You might guess that the external current the gravitational field couples to is nothing but the stress energy tensor $T^{\mu\nu}$. Use $k_\mu T^{\mu\nu}(k) = 0$ and remember that T^{00} is the energy density to show that (massive) gravity is universally attractive. **2 pts.**

Although you see that the limit of the Yukawa potential for $m \to 0$ is just the Coulomb potential it is not trivially true that the analysis using massive force particles can be extrapolated to the massless case. In fact, the coefficient of the potential changes for massless gravity due to the reduced degrees of freedom. Fortunately the force remains universally attractive.

[1] Remember exercise sheet 6 from QFT1?