Physikalisches Institut
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Theoretische Physik

Exercise sheet 8
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Exercises Quantum Field Theory II

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http://www.th.physik.uni-bonn.de/klemm/qft2ws1516/

–Homework–

1 (non-)Abelian gauge invariance (20 pts.)

You learned that electromagnetism arises from the exchange of spin-1 bosons and that the way in which the bosons couple to matter is dictated by gauge invariance. You also saw that gauge invariance enforces relations between UV-divergences and in fact these are necessary for the theory to be renormalizable. In this exercise sheet we want to take a deeper look at the QED Lagrangian and generalise its structure to non-Abelian gauge theories which underlie the strong and electroweak force.

1. The QED Lagrangian is given by

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \bar{\psi}(iD\!\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

with $D_\mu = \partial_\mu + ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Furthermore, gauge transformations act as

$$\begin{aligned} \psi(x) \to& U(x)\psi(x) = e^{i\alpha(x)}\psi, \\ A_\mu(x) \to& U(x)A_\mu(x)U(x)^{-1} - \frac{i}{e}U(x)\partial_\mu U(x)^{-1} = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x). \end{aligned} \tag{2}$$

Show that $F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu]$ and that the field strength tensor transforms as

$$[D_\mu, D_\nu] \to U(x)[D_\mu, D_\nu]U^\dagger(x). \tag{3}$$

Show also that under gauge transformations $D_\mu\psi \to U(x)D_\mu\psi$. **3 pts.**

To generalise this structure we first have to take a look at Lie groups. A *Lie group* G is roughly a group which at the same time is a differentiable manifold. The manifold has an origin which is the identity element of the group. The tangent space at the identity is called the *Lie-algebra*. In particular all the elements in the component connected to the identity can be obtained by exponentiating elements of the Lie-algebra. We concentrate on the Lie-group $SU(N)$,

$$SU(N) := \{U \in \mathbb{C}^{N\times N} \,|\, U^\dagger = U^{-1}, \det U = 1\}^1. \tag{4}$$

Every element $g \in SU(N)$ is connected by a path to the identity and can be written in the form $g = e^{i\alpha^a t_a}$, with traceless hermitian matrices t_a. The latter generate the Lie-algebra and satisfy

$$[t_a, t_b] = if^{abc}t_c, \tag{5}$$

with so-called *structure constants* f^{abc}. Note that for $SU(N)$ with hermitian generators the structure constants are real. We normalize the generators via $\mathrm{Tr}(t_a t_b) = -\frac{1}{2}\delta_{ab}$.

2. Assume your Dirac field transforms as $\psi(x) \to V(x)\psi(x)$ with $V : \mathbb{R}^{1,3} \to SU(N)$. In order to construct an invariant Lagrangian we have to introduce the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu(x), \quad A_\mu(x) = A^a_\mu(x)t_a. \tag{6}$$

Show that $\bar{\psi}(i\not{D} - m)\psi$ stays invariant if $A_\mu(x)$ transforms as

$$A_\mu(x) \to U(x)A_\mu(x)U(x)^{-1} + \frac{i}{g}U(x)\partial_\mu U(x)^{-1}. \tag{7}$$

3 pts.

3. Define $F_{\mu\nu} = F^a_{\mu\nu}t_a = \frac{i}{g}[D_\mu, D_\nu]$. Show that

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + gf^{abc}A^b_\mu(x)A^c_\nu(x) \tag{8}$$

and that $[D_\mu, D_\nu]$ transforms as

$$[D_\mu, D_\nu] \to U(x)[D_\mu, D_\nu]U^{-1}(x). \tag{9}$$

Furthermore show that $\mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu})$ is gauge invariant and that

$$\frac{1}{2}\mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{4}F^{a,\mu\nu}F^a_{\mu\nu}. \tag{10}$$

6 pts.

You have shown that the so-called Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = \bar{\psi}(i\not{D} - m)\psi + \frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{a,\mu\nu}F^a_{\mu\nu} \tag{11}$$

is invariant under local $SU(N)$ gauge transformations.

4. Show that an infinitesimal transformation (i.e. to first order in $\alpha^a(x)$) acts as

$$\begin{aligned}\psi \to& (1 + i\alpha^a t_a)\psi,\\ A^a_\mu \to& A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c,\\ F^a_{\mu\nu} \to& F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}.\end{aligned} \tag{12}$$

Note that for $U(1)$ all structure constants vanish because there is only one generator $t = 1$.
3 pts.

5. Show the Bianchi identity

$$D_\mu F^a_{\nu\rho} + D_\nu F^a_{\rho\mu} + D_\rho F^a_{\mu\nu} = 0. \tag{13}$$

Hint: The structure constants obey the Jacobi identity

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0. \tag{14}$$

5 pts.