Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 1
16. Oktober 2009
WS10/11

Exercises Superstring theory

Prof. Dr. Albrecht Klemm, Priv.-Doz. Stefan Förste

Some general remarks

Responsible for the exercises are Marco Rauch and Thomas Wotschke.
You may reach us

- via email `rauch@th.physik.uni-bonn.de` or `wotschke@th.physik.uni-bonn.de`,
- via phone `73-2549`,
- or in our office, room 104.2 in the Physikalisches Institut.

If you have any questions, remarks, need for further discussion etc. concerning the exercises or the lectures, we highly advice you to consult us during our office hour on `Wednesday` from `5-6 pm`. Feedback is also highly welcome at any time.
The lecture times are

- Monday 10-12 and Tuesday 12-13 at SR II HISKP.

You can find informations about the lecture and the exercises on the webpage
`http://www.th.physik.uni-bonn.de/klemm/strings1011/index.php`

1 Compactification and Symmetries

In this exercise we want to study two important concepts in string theory: the idea of compactification and the rise of new symmetries, which are genuine in string theory. For this we want to recap some notions in conformal field theory from last semester's course, where we especially emphasize the notion of the partition function.

1.1 Warm-Ups

1. What is the definition of the partition function and what is its physical interpretation ?

2. What are the symmetries of the partition function? Sketch the explanation from first principles. *Hint: Why is the partition function modular?*

3. Explain briefly, why in the case of compactification on a circle S^1_R the operators L_0 and $\bar{L}_0$ take the following form

$$L_0 = \frac{\alpha'}{4} p_L^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n, \quad \bar{L}_0 = \frac{\alpha'}{4} p_R^2 + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} \bar{\alpha}_{-n}\bar{\alpha}_n. \tag{1.1}$$

   What are the expressions for the zero modes p_L and p_R? Interpret the corresponding quantities!

4. Calculate the partition function of a boson on S^1_R! Can you think of any check of your result? What happens under modular transformations? *Hint: The η-function can be represented as a product $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ and it has the following transformation properties $\eta(\tau+1) = e^{\frac{\pi i}{12}}\,\eta(\tau)$, $\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$. You may also want to use the Poisson resummation formula* $\sum_{m=-\infty}^{\infty} \exp(-\pi a m^2 + 2\pi i b m) = a^{-\frac{1}{2}} \sum_{w=-\infty}^{\infty} \exp(\frac{-\pi(w-b)^2}{a})$.

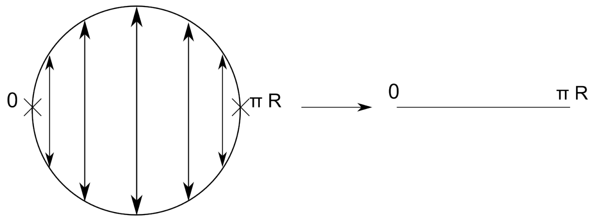
5. What is understood under the notion of T-duality?

1.2 Orbifold Compactifications

General remark
Orbifold compactifications provide interesting geometries for string compactifications. In the case of the $E_8 \times E_8$ or $SO(32)$ heterotic string, which are possible superstring theories, the compactification on orbifolds[1] allows us to make contact with the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ and to reduce the amount of supersymmetry. Furthermore this is also a subject of intensive research here in Bonn.

Exercise: Partition function of a boson on $S^1_R/\mathbb{Z}_2$
We want to consider the simplest example of an orbifold, which is given by a boson compactified on $S^1_R/\mathbb{Z}_2$. Here the $\mathbb{Z}_2$ symmetry acts via g on X as follows

$$g : X \mapsto -X,$$



The orbifold $S^1_R/\mathbb{Z}_2$ has two fixed points 0 and πR.

Our aim is to calculate the partition function of the free boson on the orbifold $S^1_R/\mathbb{Z}_2$.

1. Consider the Hilbert space $H_{S^1_R/\mathbb{Z}_2} = \{|N_L, N_R, m, n\rangle\}$ where we denote by N_L, N_R the number of oscillators, m is the momentum number and n is the winding number. Show, that the action of g on a general state is given by
$$g\,|N_L, N_R, m, n\rangle = (-1)^{N_L+N_R}\,|N_L, N_R, -m, -n\rangle\,. \tag{1.2}$$

2. Given the action of the orbifold, the Hilbert space $\mathcal{H}_{S^1_R/\mathbb{Z}_2} = \{|N_L, N_R, m, n\rangle\}$ splits into odd and even states under the action of the orbifold.
$$\mathcal{H}_{S^1_R/\mathbb{Z}_2} = \mathcal{H}_{\text{odd}} \oplus \mathcal{H}_{\text{even}}. \tag{1.3}$$
Give an explicite expression for the odd and for the even states.

3. In order to project on the invariant states under the action of the orbifold, we have to insert the projector $\frac{1+g}{2}$ into the partition function. Therefore, as a first step, we want to calculate the partition function
$$Z(\tau,\bar\tau) = \mathrm{Tr}\frac{1+g}{2} q^{L_0-\frac{c}{24}}\bar q^{\bar L_0-\frac{c}{24}} \tag{1.4}$$

[1]Where one would compactify e.g. on $T^6/\mathbb{Z}_k$

Express the results by the Jacobi theta functions $\vartheta\left[\begin{array}{c} a \\ b \end{array}\right]$ and the η function. Is this expression modular invariant?
Hint: We define the theta function with characteristics by

$$\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-a)^2} e^{2\pi i(z-b)(n-a)}. \tag{1.5}$$

The well known Jacobi theta functions are then given as

$$\begin{aligned} \vartheta_1(\tau,z) = \vartheta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](\tau,z), \quad \vartheta_2(\tau,z) = \vartheta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](\tau,z) \\ \vartheta_3(\tau,z) = \vartheta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](\tau,z), \quad \vartheta_4(\tau,z) = \vartheta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right](\tau,z) \end{aligned} \tag{1.6}$$

Under modular transformations the theta functions transform as

$$\begin{aligned} \vartheta\left[\begin{array}{c} a \\ b \end{array}\right](\tau+1,z) &= e^{i\pi a(1-a)}\vartheta\left[\begin{array}{c} a \\ a+b-\frac{1}{2} \end{array}\right](\tau,z), \\ \vartheta\left[\begin{array}{c} a \\ b \end{array}\right](-\frac{1}{\tau},\frac{z}{\tau}) &= \sqrt{-i\tau}e^{2\pi iab+i\pi\frac{z^2}{\tau}}\vartheta\left[\begin{array}{c} b \\ -a \end{array}\right](\tau,z). \end{aligned} \tag{1.7}$$

You may check this for yourself as a homework exercise. Furthermore there exist the product represenations

$$\begin{aligned} \vartheta\left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right](\tau,z) &= 2q^{\frac{1}{8}}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2\pi iz})(1-q^n e^{-2\pi iz}), \\ \vartheta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](\tau,z) &= 2q^{\frac{1}{8}}\cos(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1+q^n e^{2\pi iz})(1+q^n e^{-2\pi iz}), \\ \vartheta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n+\frac{1}{2}} e^{2\pi iz})(1+q^{n+\frac{1}{2}} e^{-2\pi iz}), \\ \vartheta\left[\begin{array}{c} 0 \\ 1/2 \end{array}\right](\tau,z) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n+\frac{1}{2}} e^{2\pi iz})(1-q^{n+\frac{1}{2}} e^{-2\pi iz}). \end{aligned} \tag{1.8}$$

Though, we will say more about modular forms in general, you may wish to consult the article by D. Zagier in [1].

4. As you should have seen by now, the partition function of the invariant states is not a modular invariant object. From this we conclude, that something is missing. Now there are two possible ways to proceed: we either demand modularity and from this we construct the modular invariant partition function. How does the modular invariant partition function look like?

5. However this raises the question, whether there are some extra states in the theory. These states are the so calles *twisted* states and they satisfy

$$X(\sigma+2\pi) = gX(\sigma) = -X(\sigma) \tag{1.9}$$

Show that a field X in the twisted sector is subject to the following expression

$$X = x + \sum_{n \in \mathbb{Z}+\frac{1}{2}} \frac{1}{n} \alpha_n z^{-n} + \bar{\alpha}_n \bar{z}^{-n} \tag{1.10}$$

6. Due to the two fixed points under the orbifold action, the vacuum energy gets shifted by $\frac{1}{16}$. Now compute in similar fashion the partion function in the twisted sector. Does this agree with you expectation?

Conclusion
We encoutered a typical example of a orbifold. In general an orbifold is given by a manifold M/G, where G is a discrete group. For the partition function we have to insert all group elements $g \in G$ into the trace taken over all twisted and untwisted sectors

$$Z = \frac{1}{|G|} \sum_{h,g \in G} \mathrm{Tr}_{\mathcal{H}}\, g\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}, \tag{1.11}$$

where we denote by $|G|$ the order of the group and a state in the h-th twisted sector satisfies

$$X(\sigma + 2\pi) = hX(\sigma) \tag{1.12}$$

Brain teaser: Shift symmetries
We could also realize a different $\mathbb{Z}_2$ symmetry by shifting the coordinates as

$$X \mapsto X + \pi R. \tag{1.13}$$

Calculate the partition function of this orbifold! *Hint: You may seek help in [2].*

References

[1] Bruinier, J.H. and Harder, G. and van der Gerr, G. and Zagier, D, “The 1-2-3 of modular forms,” Springer Berlin Heidelberg, 2008

[2] E. Kiritsis, “String theory in a nutshell,”Princeton, USA: Univ. Pr. (2007) 588 p