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Theoretische Physik

Exercise 2
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Exercises Superstring Theory

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1 The classical superstring

In this exercise we want to study the closed, classical superstring. We consider manifest $N = 1$ supersymmetry on the world-sheet (σ, τ). On-shell the field content is given by a world-sheet scalar $X^\mu(\sigma, \tau)$ and a world-sheet Majorana spinor $\psi^\mu(\sigma, \tau)$, which are both vectors of the d-dimensional target space. Written in superconformal gauge and analyzed in light-cone coordinates on the world-sheet, $\sigma^\pm = \tau \pm \sigma$, the action reads

$$S = \frac{1}{2\pi} \int d^2\sigma \big(\partial_+ X \cdot \partial_- X + i(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-) \big). \tag{1.1}$$

1.1 Warm-Ups

1. Write down the equations of motions for the action (1.1).
2. The above equations of motions have to be supplemented by periodicity conditions. To what refer Ramond (R) and Neveu-Schwarz (NS) boundary conditions?
3. What is the conformal weight of ψ?

1.2 Super-de-Witt algebra

Due to the superconformal symmetry in two dimensions we have an infinite number of conserved (super)charges, which are encoded in the energy-momentum tensor T_{ab} and its supersymmetric partner G_a given by

$$\begin{aligned} T_{\pm\pm} &= \frac{1}{2} \partial_\pm X \cdot \partial_\pm X + \frac{i}{2} \psi_\pm \cdot \partial_\pm \psi_\pm, \quad T_{+-} = T_{-+} = 0, \\ G_\pm &= \frac{1}{2} \psi_\pm \cdot \partial_\pm X. \end{aligned} \tag{1.2}$$

1. Write down the mode expansions for X^μ and for $\psi^\mu_\pm$ for both the R and NS sector. Into what condition on the modes b^μ_r, $\bar{b}^\mu_r$ of $\psi^\mu_\pm$ does the reality of the Majorana spinors translate?
2. Using the classical brackets (we only display the non-vanishing ones)

$$\begin{aligned} \{X^\mu(\sigma, \tau), \dot{X}^\nu(\sigma', \tau)\}_{\text{cl}} &= -4\pi i \delta(\sigma - \sigma') \eta^{\mu\nu}, \\ \{\psi^\mu_\pm(\sigma, \tau), \psi^\nu_\pm(\sigma', \tau)\}_{\text{cl}} &= -2\pi i \delta(\sigma - \sigma') \eta^{\mu\nu}, \end{aligned} \tag{1.3}$$

   deduce the commutation relations for the modes[1]

$$\begin{aligned} \{\alpha^\mu_m, \alpha^\nu_n\}_{\text{cl}} = \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\}_{\text{cl}} &= -im\delta_{m+n} \eta^{\mu\nu}, \\ \{b^\mu_r, b^\nu_s\}_{\text{cl}} = \{\bar{b}^\mu_r, \bar{b}^\nu_s\}_{\text{cl}} &= -i\delta_{r+s} \eta^{\mu\nu}. \end{aligned} \tag{1.4}$$

[1] b^μ_r, $\bar{b}^\mu_r$ denote the modes of $\psi^\mu_\pm$; α^μ_n, $\bar{\alpha}^\mu_n$ are the modes of X^μ.

3. Define

$$L_m = \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{-im\sigma} T_{--}, \quad G_r = \frac{1}{\pi}\int_0^{2\pi} d\sigma e^{-ir\sigma} G_{-}. \tag{1.5}$$

What is L_m and G_r in terms of the modes α_n and b_r?

4. Show, that the modes of the energy momentum-tensors satisfy the supersymmetric extension of the loop algebra/ de-Witt algebra

$$\begin{aligned}\{L_m, L_n\}_{\text{cl}} &= -i(m-n)L_{m+n},\\ \{L_m, G_r\}_{\text{cl}} &= -i(\frac{m}{2}-r)G_{m+r},\\ \{G_r, G_s\}_{\text{cl}} &= -2iL_{r+s}.\end{aligned} \tag{1.6}$$

2 Take home problem: The quantized superstring

In this second exercise we want to move on to the quantized superstring and compute the quantum version of the algebra satisfied by L_m and G_r, the so-called $N=1$ superconformal algebra (SCA).

1. How do the commutation relations (1.4) look after canonical quantization?

2. As operator ordering prescription we use as in the bosonic case normal ordering, which means that we put negative frequency modes to the right of positive frequency modes. E.g., $:\alpha_m\alpha_{-m} := \alpha_{-m}\alpha_m$ for $m \geq 0$. We denote the yet undetermined normal ordering constant by a. Now show, that L_m and G_r satisfy the SCA

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{8}m(m^2-2a)\delta_{m+n}, \tag{2.7}$$

where $a=0$ and $a=\frac{1}{2}$ for the R and NS sectors, respectively. Analoguously, one can derive

$$\begin{aligned}[L_m, G_r] &= (\frac{m}{2}-r)G_{m+r},\\ \{G_r, G_s\} &= 2L_{r+s} + \frac{d}{2}(r^2-\frac{a}{2})\delta_{r+s}.\end{aligned} \tag{2.8}$$

3. Concentrate on the NS sector for now. Using a ζ-function regularization show, that the normal ordering constant a is related to the dimension of space-time d by $a=\frac{d-2}{16}$. What is the critical dimension of the superstring?

Hint: The ζ-function is defined by

$$\zeta(s,a) = \sum_{n\geq 0}\frac{1}{(n+a)^s}, \tag{2.9}$$

and we have the following general formula $\zeta(-1,a) = -\frac{1}{12}(6a^2-6a+1)$.

References

If you want to read more on the subject of today's class, you may consult e.g.:

D. Lüst, S. Theisen, "Lectures on String Theory", chapter 7 and 8.