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Exercise 3
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Exercises Superstring theory

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1 Spinors in various dimensions

In this exercise we want to study the Clifford algebra and its representations in various dimensions. This is important as the superstring forces us to have a ten dimensional space-time.

1.1 Warm-Ups

1. What is the definition of the Clifford algebra?
2. For the case of $d = 2$ can you write down a representation of the Clifford algebra?
3. Now we turn to something more technical: denote by A a $m \times n$ matrix and by B a $p \times q$ matrix. What is the definition of the tensor product $A \otimes B$?
4. Show the following properties

$$\begin{aligned} (A \otimes B)^T = A^T \otimes B^T, (A \otimes B)^* = A^* \otimes B^* \\ (A \otimes B)(C \otimes D) = (AC \otimes BD), \end{aligned} \tag{1.1}$$

Now we consider matrices A of type $m \times m$ and matrices B of type $n \times n$, then show

$$\det(A \otimes B) = (\det A)^n (\det B)^m, \ \mathrm{Tr}(A \otimes B) = \mathrm{Tr}A\mathrm{Tr}B \tag{1.2}$$

1.2 Recursive construction of representations of the Clifford algebra in various dimensions

General remark
We want to study a recursive construction of representations of the Clifford algebra in various dimensions. For this we start in $d = 2$ dimensions, where the two Γ matrices read

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

for the case of even dimensions, we denote the $d-2$ dimensional matrices by γ^μ and then the d dimensional matrices are given by

$$\Gamma^\mu = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \gamma^\mu, \quad \Gamma^{d-2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1}, \qquad \Gamma^{d-1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \mathbb{1} \tag{1.4}$$

For the case that d is odd, we add the matrix

$$\Gamma_* = i^\alpha \Gamma^0 \cdot \dots \cdot \Gamma^{d-1}, \tag{1.5}$$

where α is a parameter such that if d is even we have

$$\alpha = \begin{cases} 1 & d \equiv 0 \mod 4 \\ 0 & d \equiv 2 \mod 4 \end{cases}, \tag{1.6}$$

and for odd d we have $\alpha = \frac{d+1}{2} \mod 2$.

Exercises

1. Show that this procedure defines a representation of the Clifford algebra in d dimensions.
2. Show that, in any dimensions, Γ^0 and all odd Γ^i for $i \geq 3$ are antisymmetric, while Γ^1 and the even Γ^i are symmetric. Conclude that $\Gamma^3, \Gamma^5, \ldots, \Gamma^9$ are imaginary and the other ones are real.
3. Now we consider a Majorana condition, i.e. a reality condition on a spinor of the form $\psi^* = B\psi$. The corresponding Lorentz generator is given by

   $$\Sigma^{\mu\nu} = \frac{i}{4} [\Gamma^\mu, \Gamma^\nu] \tag{1.7}$$

   and the Majorana condition requires

   $$B\Sigma^{\mu\nu}B^{-1} = -\Sigma^{\mu\nu}, \qquad \text{and} B^*B = 1. \tag{1.8}$$

   Again we start with even dimensions and we define

   $$B = \Gamma^3\Gamma^5 \cdots \Gamma^{D-1}, \qquad B' = \Gamma_* B \tag{1.9}$$

   Show that

   $$B\Gamma^\mu B^{-1} = -(-1)^{\frac{D}{2}}(\Gamma^\mu)^*, \qquad B'\Gamma^\mu B'^{-1} = (-1)^{\frac{D}{2}}(\Gamma^\mu)^*, \tag{1.10}$$

   so B and B' satisfy the first condition from above. For which d do they also satisfy the second one? Under which condition is the Majorana condition compatible with a chirality condition $\Gamma_*\psi = \pm\psi$? The definitions of B and B' also extend to $d+1$ dimensions. Do they both generate consistent Majorana conditions? In which dimensions?

1.3 The GSO Projection

In supersymmetric string theories one encounters several problems, like tachyonic states and non supersymmetric spectra. However, these problems can be solved by introducing the GSO projection. First one introduces the concept of G parity, which is given in the NS sector by

$$G = (-1)^{F+1} = (-1)^{\sum_{r=\frac{1}{2}}^{\infty} b^i_{-r} b^i_r + 1}, \tag{1.11}$$

which determines the number of world-sheet fermion excitations. In the R sector the G parity operator is given by

$$G = \Gamma_*(-1)^{\sum_{n=1}^{\infty} d^i_{-n} d^i_n} \tag{1.12}$$

The GSO projection now only projects on states in the NS sector with positive G parity, i.e.

$$(-1)^{F_{\rm NS}} = -1, \tag{1.13}$$

where in the R sector one either keeps states with positive or negative parity.

Exercises

1. Show that the tachyon is projected out.
2. Calculate the generating functions that encode the number of physical ddegrees of freedom in the NS and R sectors at all levels after GSO projection.

Take-Home exercise

1. Write down and discuss the spectra of type IIA and IIB string theory! What is the difference between these two theories?
2. Show that there are the same number of physical degrees of freedom in the NS and R sectors at the first massive level after the GSO projection. This fact enforces modularity of the partition function, which you are asked to calculate!

1.3.1 Literature

Concerning the spectrum, you may find help in the book by Lüst and Theisen on string theory. Concerning spinors in various dimensions, you may want to check the appendix of Polchinski's book on superstring theory.
In case you are interested in more mathematical aspects of Clifford algebras, you may want to have a look at the book "Spin Geometry" by Michelson and Lawson.