Exercises Superstring Theory

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1 Spin structures and modular transformations

Let Σ_g be a genus g Riemann surface. In order to define spinors on Σ_g, we have to assign to them either periodic or anti-periodic boundary conditions around each a- and b-cycle. Each possiblity of these assignments is called a spin structure.

1. How many spin structures are there on Σ_g?

In the following let's assume $g = 1$. Complex coordinates on the torus are given by $z = \xi^1 + \tau\xi^2$ with $\xi^i \in [0, 1]$ and τ parameterizes the different complex structures. A flat metric is given by $ds^2 = |dz|^2$ and thus the Dirac operator is simply ∂_z.

2. How many even and odd spin structures on the torus are there?
3. Under a general modular transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, how does the metric ds^2 change?
4. Work out how the different possible boundary conditions of fermions transform under S and T.

2 Superstring partition function

In the following we want to work out the vacuum amplitude on the torus, i.e. the partition function, for the superstring. We will work in light-cone gauge.

2.1 Warm-up: Bosonic strings

1. Derive once more the partition function of bosonic string theory.
2. Show, that the bosonic partition function is modular invariant.

2.2 Superstrings

1. Show that the (right-moving) Hamiltonians for the R and NS sector, H_{R} and H_{NS}, are given by

$$\begin{aligned} H_{\text{R}} &= \sum_{m=1}^{\infty} m b^i_{-m} b^i_m + \frac{1}{3}, \\ H_{\text{NS}} &= \sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r} b^i_r - \frac{1}{6}. \end{aligned} \tag{2.1}$$

2. Compute the contributions to the partition functions from the different spin structures. You should obtain

$$
\begin{aligned}
Z^{(--)}(\tau) &= \varphi^{(--)} \mathrm{Tr}\, q^{H_{\mathrm{NS}}} = \varphi^{(--)} \frac{\vartheta_3^4(\tau)}{\eta^4(\tau)}, \\
Z^{(-+)}(\tau) &= \varphi^{(-+)} \mathrm{Tr}\, (-1)^F q^{H_{\mathrm{NS}}} = \varphi^{(-+)} \frac{\vartheta_4^4(\tau)}{\eta^4(\tau)}, \\
Z^{(+-)}(\tau) &= \varphi^{(+-)} \mathrm{Tr}\, q^{H_{\mathrm{R}}} = \varphi^{(+-)} \frac{\vartheta_2^4(\tau)}{\eta^4(\tau)}, \\
Z^{(++)}(\tau) &= \varphi^{(++)} \mathrm{Tr}\, (-1)^F q^{H_{\mathrm{R}}} = \varphi^{(++)} \frac{\vartheta_1^4(\tau)}{\eta^4(\tau)},
\end{aligned}
\tag{2.2}
$$

where φ are phases to be determined by modular invariance and the entries $(\pm, \pm)$ refer to the boundary conditions along the two cycles of the torus.

3. By requiring modular invariance of the partition function determine the phases φ. How do you see that the spectrum is supersymmetric? Show, that $\vartheta_1(\tau) = 0$.

Some modular forms

Definition of Jacobi theta-functions:

$$\vartheta_1(\tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} (-1)^n q^{\frac{1}{2}n^2}, \tag{2.3}$$

$$\vartheta_2(\tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{\frac{1}{2}n^2}, \tag{2.4}$$

$$\vartheta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2}, \tag{2.5}$$

$$\vartheta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2}. \tag{2.6}$$

They have the product representations

$$\vartheta_1(\tau) = i\eta(\tau) q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1}), \tag{2.7}$$

$$\vartheta_2(\tau) = 2\eta(\tau) q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 + q^n)^2, \tag{2.8}$$

$$\vartheta_3(\tau) = \eta(\tau) q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})^2, \tag{2.9}$$

$$\vartheta_4(\tau) = \eta(\tau) q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}})^2. \tag{2.10}$$

Modular transformations of $\vartheta_i(\tau)$:

$$\vartheta_2(-1/\tau) = \sqrt{\frac{\tau}{i}} \vartheta_4(\tau), \qquad \vartheta_2(\tau + 1) = e^{i\pi/4} \vartheta_2(\tau), \tag{2.11}$$

$$\vartheta_3(-1/\tau) = \sqrt{\frac{\tau}{i}} \vartheta_3(\tau), \qquad \vartheta_3(\tau + 1) = \vartheta_4(\tau), \tag{2.12}$$

$$\vartheta_4(-1/\tau) = \sqrt{\frac{\tau}{i}} \vartheta_2(\tau), \qquad \vartheta_4(\tau + 1) = \vartheta_3(\tau). \tag{2.13}$$

There is the identity

$$\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau). \tag{2.14}$$

Definition of the eta-function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.15}$$

Modular transformations of η:

$$\eta(\tau + 1) = e^{i\pi/12}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau). \tag{2.16}$$