Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 5
November 29th, 2010
WS10/11

Exercises Superstring theory

Prof. Dr. Albrecht Klemm, Priv.-Doz. Dr. Stefan Förste

1 Modularity

The aim of this exercise is to give some idea what is understood by the notion of modularity. We have encountered the notion of modularity in the context of partition functions and we already used some techniques. We want to prove some identities.

1.1 Warm-Ups

1. Once again: please recall why the partition function is modular invariant!

1.2 Definition and Identities

We want to consider modular transformations of the form

$$\tau \mapsto \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \gamma \in SL(2, \mathbb{Z}). \tag{1.1}$$

Now we want to start with the most general definition which is convenient for our needs, namely Jacobi forms. A *Jacobi form* of weight $k \in \mathbb{N}_0$ and index $m \in \mathbb{N}_0$ is an analytic function

$$f(\tau, \nu) : \mathcal{H} \times \mathbb{C} \to \mathbb{C}, \tag{1.2}$$

which for $(\lambda, \mu) \in \mathbb{Z}^2$ satisfies

$$\begin{aligned} f\left(\gamma(\tau), \frac{\nu}{c\tau + d}\right) &= (c\tau + d)^k \exp\left(\frac{2\pi i m c \nu^2}{c\tau + d}\right) f(\tau, \nu), \\ f(\tau, \nu + \lambda\tau + \mu) &= \exp(2\pi i m(-\lambda^2 \tau - 2\lambda\nu)) f(\tau, \nu). \end{aligned} \tag{1.3}$$

Furthermore one demands at most polynomial growth of $f(\tau, \nu)$ at the cusps, i.e. at $\tau \to i\infty$. Then, this is equivalent to the existence of a Fourier expansion given by

$$f(\tau, \nu) = \sum_{n=0}^{\infty} \sum_{r \in \mathbb{Z}} c(n, r) q^n z^r, \tag{1.4}$$

for $4nm - r^2 \geq 0$ and $q = e^{2\pi i \tau}, z = e^{2\pi i \nu}$. A modular form of weight k is a holomorphic function $f(\tau)$ on $\mathcal{H}$ satisfying (1.3) with index $m \equiv 0$.

As a first example we define the basic theta function given by its Fourier expansion

$$\theta(\nu, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n \nu) = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} z^n \tag{1.5}$$

Exercises

1. Analyse its periodicity for $\nu \mapsto \nu + 1$ and $\nu \mapsto \nu + \tau$!
2. Prove the Poisson resummation formula for f and its Fourier transform $\hat{f}$,

$$\sum_{n\in\mathbb{Z}} f(x + nT) = \frac{1}{T} \sum_{k\in\mathbb{Z}} \hat{f}\left(\frac{k}{T}\right) e^{\frac{2\pi i k x}{T}} \tag{1.6}$$

3. Apply the Poisson resummation to analyze the modularity properties of (1.5),

$$\theta(\nu, \tau + 1) = \theta(\nu + \frac{1}{2}, \tau), \quad \theta\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\pi i \frac{\nu^2}{\tau}} \theta(\nu, \tau) \tag{1.7}$$

4. Rewrite $\theta(\nu, \tau)$ as an infinite product using Jacobi's triple product identity,

$$\prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}} t)(1 + q^{n-\frac{1}{2}} t^{-1}) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} t^n. \tag{1.8}$$

   What is the behavior for $q \to 0, q \to 1$ and $z \to -q$?

The theta function with characteristics is defined by

$$\vartheta \left[\begin{array}{c} a \\ b \end{array} \right] (\nu, \tau) = \exp(\pi i a^2 \tau + 2\pi i a(\nu + b))\theta(\nu + a\tau + b, \tau). \tag{1.9}$$

that are also denoted by

$$\begin{aligned} \vartheta \left[\begin{array}{c} 0 \\ 0 \end{array} \right] (\nu, \tau) = \theta_3(\nu, \tau), \quad \vartheta \left[\begin{array}{c} 0 \\ 1/2 \end{array} \right] (\nu, \tau) = \theta_4(\nu, \tau) \\ \vartheta \left[\begin{array}{c} 1/2 \\ 0 \end{array} \right] (\nu, \tau) = \theta_2(\nu, \tau), \quad \vartheta \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (\nu, \tau) = \theta_1(\nu, \tau) \end{aligned} \tag{1.10}$$

1. Determine the sum as well as the product representation of (1.10).
2. What are the periodicity properties for $\nu \mapsto \nu + 1$ and $\nu \mapsto \nu + \tau$?
3. Analyze the modularity behavior!
4. Evaluate the theta functions at $\nu = 0$ and deduce the behavior under modular transformations.

The Dedekind eta-function is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{1.11}$$

1. Show the relation

$$\eta^3(\tau) = \frac{1}{2} \theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \tag{1.12}$$

   and infer the transformation property under modular transformations.
2. Finally compare all encountered examples and read of the weight and the index.