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Exercises Superstring Theory

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1 Quantization and Dirac brackets

The usual route to quantum field theory involves the classical description of a system giving a Lagrangian, find the Hamiltonian and quantize the Hamiltonian. There are, however, complications when one uses the standard Hamiltonian formulation.

1.1 Warm-Up

1. Explain briefly the Lagrange formulation of classical mechanics. What are the action, the Lagrangian and canonical momenta?

2. Review briefly the standard Hamiltonian method. What are Poisson brackets and what are their properties?

1.2 A Lagrangian linear in velocities

Consider a classical, charged particle confined to the x-y plane subject to a constant magnetic field in the z-direction and some external potential $V(x, y)$. The Lagrangian can be written as

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \eta(x\dot{y} - y\dot{x}) - \eta V(x, y). \tag{1.1}$$

1. Find the equations of motion.

2. In the large magnetic field limit $\eta \to \infty$ which terms in L can be neglected? Write down the new Lagrangian L' and its equations of motion.

3. Starting from L', what are the canonical momenta? What is the Hamiltonian and what are Hamilton's equations? What do you observe?

1.3 Dirac brackets

There are certain cases in which coordinates and momenta are not independent and we have constraints between them. This happens, for instance, if the conjugate momenta are not functions of the velocities $\dot{q}_i$, in which case we get an "on-shell" relation $\phi_j(q, p) = 0$ between coordinates and momenta. In this case the standard Hamiltonian method breaks down and we cannot appropriately quantize the system using the Poisson brackets. One encounters so-called second-class constraints, that are constraints who have non-vanishing Poisson bracket with at least one other constraint. Brackets which respect the constraints are known as Dirac brackets and defined by

$$\{f, g\}_{\text{D.B.}} = \{f, g\} - \{f, \phi_k\} M^{-1}_{kl} \{\phi_l, g\}, \tag{1.2}$$

where ϕ_k are second class constraints and the brackets on the right are usual Poisson brackets. M is given by $M_{ij} = \{\phi_i, \phi_j\}$.

Further, the Hamiltonian has to be generalized by including the constraints (similar to a Lagrange multiplier)

$$H_T = H + u_j \phi_j, \tag{1.3}$$

where u_j are constants, which are subject to the consistency condition $\dot{\phi}_i = \{\phi_i, H_T\} = 0$, which is valid after imposing the equations of motion. The new Hamilton equations are now written as

$$\begin{aligned} \dot{q} &= \{q, H_T\}_{\text{D.B.}} \\ \dot{p} &= \{p, H_T\}_{\text{D.B.}}. \end{aligned} \tag{1.4}$$

1. Identify the two constraints in problem 1.2 and calculate M and H_T.
2. Compute the equations of motion using the Dirac brackets.
3. Calculate the commutation relations among the coordinates and momenta in Dirac brackets and quantize the system.

1.4 An example with fermions

Consider the Lagrangian

$$L = i\bar{\psi}\dot{\psi} - m\bar{\psi}\psi, \tag{1.5}$$

where $\psi = \psi(t)$ is a Grassmann variable.

1. Write down the equations of motion.
2. Find the canonical conjugate momenta $\pi, \bar{\pi}$. Determine M and H_T.
   Note, that $\{f, g\} = -\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + (-1)^{\epsilon_f \epsilon_g}\frac{\partial g}{\partial p}\frac{\partial f}{\partial q}$, *where* ϵ_f *is the Grassmann parity of* f.
3. Calculate all Dirac brackets among coordinates and momenta. Quantize the system.

These subtleties occur always when dealing with fermions and hence also in the RNS or Green-Schwarz formulation of superstring theory.

References

Paul Dirac – *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monographs Series Number Two (1964)