Exercises String Theory

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1 The relativistic point particle

We consider the action for a relativistic point particle

$$S_{pp} = -m \int ds = -m \int \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} d\tau, \tag{1.1}$$

where τ parameterizes the worldline of the particle.

1. Show that the action is invariant under Poincaré transformations.
2. Show that the action is invariant under reparametrizations of the worldline time $\tau \to \tau'(\tau)$.
3. Show that
$$p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\eta_{\nu\rho}\dot{X}^\nu \dot{X}^\rho}} \tag{1.2}$$
is a conserved quantity, by evaluating once the Euler Lagrange equations and once by exploiting the symmetry $X^\mu \to X^\mu + b^\mu$.
4. Why is this action inappropriate to describe massless particles?
5. Show that
$$S_e = -\frac{1}{2}\int d\tau e\Big(-\frac{1}{e^2}\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} + m^2\Big) \tag{1.3}$$
is equivalent to the action (1.1).
Hint: Integrate out e.
6. Explain the statement "We have coupled the particle to worldline gravity". What kind of field is e?
7. Show the invariance of the new action (1.3) under reparametrizations of τ, how does e transform?

2 The Nambu-Goto action versus Polyakov action

The Nambu-Goto action for a string is given by

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)}. \tag{2.1}$$

Here σ^α, $\alpha = 0, 1$ label the worldsheet time τ and space σ.

1. Write down explicitly the action (2.1), i.e. without refering to α, but to 0,1 instead.
2. Show that the Polykov action

$$S_p = -\frac{T}{2}\int d^2\sigma\sqrt{-h}\big(h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\big) \tag{2.2}$$

is equivalent to the Nambu-Goto action.

Homework **10 Points**

3 Symmetries of the string and their implications **5 Points**

3.1 Global symmetries **2 Points**

Show that the action is invariant under Poincaré transformations

$$X^\mu \to \Lambda^\mu_\nu X^\nu + b^\mu. \tag{3.1}$$

Evaluate the corresponding conserved currents using the Noether procedure, which we briefly recall. If the Lagrangian is invariant under an infinitesimal transformation of the fields given by

$$\phi^a \to \phi^a + \delta\phi^a, \quad \delta\phi^a = \epsilon^i h_i^a(\phi^b), \tag{3.2}$$

where ϵ^i is infinitesimal and h_i^a denotes a function of the fields ϕ^a, then the current j_i^α defined by

$$\epsilon^i j_i^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\delta\phi^a \tag{3.3}$$

is conserved. Note that i might be a multi-index. The infinitesimal variations for Poincaré transformations are respectively given by

$$X^\mu \to X^\mu + \epsilon^\mu, \quad X_\mu \to \epsilon a_{\mu\nu}X^\nu, \quad a_{\mu\nu} = -a_{\nu\mu}. \tag{3.4}$$

Hint: You may evaluate the currents using

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

See also the exercise (3.2) and equation (5.2).

3.2 Local symmetries **3 Points**

1. Show that the action is invariant under worldsheet reparametrizations

$$\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta). \tag{3.6}$$

2. Show that the action is invariant under Weyl transformations

$$h_{\alpha\beta} \to e^{\phi(\sigma^\alpha)}h_{\alpha\beta}. \tag{3.7}$$

3. Show that the local symmetries can be used to fix the world-sheet metric locally to

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.8}$$

Why is it not possible to choose this metric globally?

4 Two-dimensional gravity **2 Points**

Show that the energy momentum tensor in two dimensions vanishes identically due to Einstein's equation. Hint: The only non-vanishing component of the Riemann tensor in two dimensions is

$$R_{0101} = -R_{0110} = -R_{1001} = R_{1010}. \tag{4.1}$$

5 The equations of motion **3 Points**

In the following, we consider the range of world-sheet coordinates given by

$$\tau \in \mathbb{R}, \quad \sigma \in [0, \pi]. \tag{5.1}$$

1. Show that for the gauge fixed metric (3.8) the action takes the simple form

$$S = \frac{T}{2} \int d^2\sigma \big(\dot{X}^2 - X'^2\big). \tag{5.2}$$

   Here we have denoted by $\dot{}$ and $'$ the derivative with respect to τ and σ respectively.

2. Derive the equations of motion. In addition, show that there is a boundary term

$$-T \int d\tau \Big(X'_\mu \delta X^\mu \Big|_{\sigma=\pi} - X'_\mu \delta X^\mu \Big|_{\sigma=0} \Big). \tag{5.3}$$

3. Show that there are three possibilities in order to make the boundary term vanish.

   a)
$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau) \tag{5.4}$$

   b)
$$X'_\mu(\sigma, \tau) = 0, \quad \sigma = 0, \pi \tag{5.5}$$

   c)
$$X^\mu \Big|_{\sigma=0} = X_0^\mu, \quad X^\mu \Big|_{\sigma=\pi} = X_\pi^\mu \tag{5.6}$$

   Comment on the physical interpretations of the three boundary conditions. Why is the last one "strange"?

The general solution to (5.2) has the form

$$X^\mu(\sigma, \tau) = f(\sigma) g(\tau). \tag{5.7}$$

4. Show that for closed strings

$$\frac{\partial^2 f(\sigma)}{\partial \sigma^2} = c f(\sigma), \quad \frac{\partial^2 g(\tau)}{\partial \tau^2} = c g(\tau), \quad c = -4m^2, \quad m \in \mathbb{Z}. \tag{5.8}$$

5. Conclude that the general solution for closed strings takes the form

$$X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma), \tag{5.9}$$

where

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)} \tag{5.10}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \tag{5.11}$$

Here n runs over all non-zero integers. Which conditions have to be imposed on α_n^μ, $\tilde{\alpha}_n^\mu$ in order to make X^μ real?