Exercises String Theory

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1 Modular invariance of the torus partition function **5 points**

We consider the modular invariance of the partition function of a closed string propagating on an n-dimensional torus T^n. The torus is specified by its metric G_{IJ} and background B-field B_{IJ}. Their inverses are as usual denoted by raised indices. This torus should not be confused with the worldsheet torus, that is specified by $\tau = \tau_1 + i\tau_2$ taking values in the fundamental domain of $SL(2,\mathbb{Z})$. The partition function itself reads

$$Tr\Big(q^{L_0}\bar{q}^{\tilde{L}_0}\Big) = Tr\Big(q^{\frac{1}{2}p_R^2}\bar{q}^{\frac{1}{2}p_L^2}\Big) = \sum_{W^I,K_I} e^{i\pi\tau_1(p_R^2-p_L^2)}e^{-\pi\tau_2(p_L^2+p_R^2)}, \quad q = e^{2\pi i\tau}. \tag{1.1}$$

Here we have used the following abbrevations

$$p_L^I = W^I + G^{IJ}\big(\frac{1}{2}K_J - B_{JK}W^K\big), \quad p_R^I = -W^I + G^{IJ}\big(\frac{1}{2}K_J - B_{JK}W^K\big), \tag{1.2}$$

where K_I and W_I denote the momentum respectively the winding mode in the $I'th$ direction. The invariance under $SL(2,\mathbb{Z})$ needs not to be checked for the partition function alone but for the path integral which results in the combination

$$(\tau_2)^{\frac{n}{2}}Tr\Big(q^{\frac{1}{2}p_R^2}\bar{q}^{\frac{1}{2}p_L^2}\Big), \tag{1.3}$$

where the first factor stems from the integration over the momentum.

1. Show the invariance under the transformation $T : \tau \to \tau + 1$
2. Show the invariance under the transformation $S : \tau \to -\frac{1}{\tau}$

Hint: For the second step you need the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \tag{1.4}$$

where $\hat{f}$ denotes the Fourier transform of f. Note that the Fourier transform of e^{-at^2} is given by $\sqrt{\frac{1}{a}}e^{-\frac{u^2}{a}}$. Use this to show that

$$\sum_{M\in\mathbb{Z}^m} e^{-M^tAM} = \frac{1}{\sqrt{\det A}}\sum_{M\in\mathbb{Z}^m} e^{-M^tA^{-1}M}, \tag{1.5}$$

where A is a symmetric $m\times m$ matrix. Show then that the A that is relevant for you is given by

$$A = \pi\tau_2\begin{pmatrix} 2(G - BG^{-1}B) & BG^{-1} \\ -G^{-1}B & \frac{1}{2}G^{-1}\end{pmatrix} + i\pi\tau_1\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \tag{1.6}$$

2 Path integral derivation of T-duality **5 points**

We consider a scalar field

$$\varphi : \Sigma_g \longrightarrow S_1, \tag{2.1}$$

where Σ_g is a Riemann surface of genus g and $\varphi = \frac{x}{R}$ has periodicity 2π, R being the radius of the circle. In addition, we choose local coordinates σ^1, σ^2. The worldsheet metric is given by $h = h_{\mu\nu} d\sigma^\mu d\sigma^\nu$. Finally we introduce a basis ω^i, $i = 1, \dots 2g$ of the second cohomology group $H^1(\Sigma_1, \mathbb{R})$ such that the Poincaré dual one-cycles γ_j, $j = 1, \dots 2g$ defined by

$$\int_{\gamma_j} \omega^i = \delta^i_j \tag{2.2}$$

form a basis of the integer valued homology group $H_1(\Sigma_g, \mathbb{Z})$. Then

$$\int_{\Sigma_g} \omega^i \wedge \omega^j = J^{ij} \tag{2.3}$$

is an element of $GL(2g, \mathbb{Z})$. The action of φ is given by

$$S_\varphi = \frac{1}{4\pi} \int_\Sigma R^2 h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sqrt{h} d^2\sigma. \tag{2.4}$$

1. Consider the action

$$S' = \frac{1}{2\pi} \int_{\Sigma_g} \frac{1}{2R^2} h^{\mu\nu} B_\mu B_\nu \sqrt{h} d^2\sigma + \frac{i}{2\pi} \int_{\Sigma_g} B \wedge d\varphi. \tag{2.5}$$

   and use the equations of motion for B to show that S' is equivalent to S. (Alternatively you may complete the square.) Hint: You should find that

$$B = iR^2 * d\varphi. \tag{2.6}$$

2. The most general form of $d\varphi$ is given by

$$d\varphi = d\varphi_0 + \sum_{i=1}^{2g} 2\pi i n_i \omega^i, \tag{2.7}$$

   where φ_0 is single-valued on Σ_g. By integrating out $d\varphi_0$ show that B has to take the form

$$B = d\vartheta_0 + \sum_{i=1}^{2g} a_i \omega^i \tag{2.8}$$

3. Integrating out the multi-valued part of $d\varphi$ is done by summing over the integers n_i. Use this and the identity

$$\sum_n e^{ian} = 2\pi \sum_m \delta(a - 2\pi m) \tag{2.9}$$

   to show that the $B = d\vartheta$, where ϑ is a variable of period 2π.

4. Plugging this into the original action, show that one ends up with

$$S_{\vartheta} = \frac{1}{4\pi} \int_{\Sigma_g} \frac{1}{R^2} h^{\mu\nu} \partial_\mu \vartheta \partial_\nu \vartheta \sqrt{h} d^2\sigma, \tag{2.10}$$

i.e. a sigma model of radius R is equivalent with one with radius $\frac{1}{R}$.

5. Identity $Rd\varphi$ and $iR * d\varphi$ with the currents of winding and momentum modes respectively. Which one is topological and which one a proper current? What happens under T-duality?