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Exercise 3
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Exercises String Theory

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1 The Quantum Virasoro Algebra

3 Points

Recall that the classical Virasoro generators read

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n. \tag{1.1}$$

We define the generators of the quantum algebra by mapping the modes to their corresponding operators and normal ordering this expression

$$\hat{L}_m =: \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{m-n} \cdot \hat{\alpha}_n \ : \ . \tag{1.2}$$

In the following we omit the hats again. Our goal is to calculate the quantum algebra

$$[L_m, L_n] = ?. \tag{1.3}$$

1. Show that
$$[L_m, \alpha_n] = -n\alpha_{m+n}. \tag{1.4}$$

2. Show that the normal ordered version of L_m reads
$$L_m = \frac{1}{2} \sum_{n=0}^{\infty} \alpha_{m-n} \cdot \alpha_n + \frac{1}{2} \sum_{n=-\infty}^{-1} \alpha_n \cdot \alpha_{m-n}. \tag{1.5}$$

3. Use this to show that
$$\begin{aligned} [L_m, L_n] \quad = \quad & \frac{1}{2} \sum_{k=0}^{\infty} (m-k)\alpha_{m+n-k} \cdot \alpha_k + \frac{1}{2} \sum_{k=-\infty}^{-1} (m-k)\alpha_k \cdot \alpha_{m+n-k} \\ & + \frac{1}{2} \sum_{k=0}^{\infty} k\, \alpha_{m-k} \cdot \alpha_{k+n} + \frac{1}{2} \sum_{k=-\infty}^{-1} k\, \alpha_{k+n} \cdot \alpha_{m-k}. \end{aligned} \tag{1.6}$$

   Is this expression normal ordered?

4. Show that in case $m + n \neq 0$ one has
$$[L_m, L_n] = (m-n)L_{m+n}, \quad m+n \neq 0. \tag{1.7}$$

Show that for $m = -n$

$$[L_m, L_{-m}] = 2mL_0 + \frac{D}{12}(m^3 - m). \tag{1.8}$$

Hint: You might find the relation

$$\sum_{k=1}^{m} = \frac{1}{6}(2m^3 + 3m^2 + m) \tag{1.9}$$

useful.

2 Physical states and spurious states **2 Points**

We start be recalling some definitions. A state $|\eta\rangle$ satisfies the mass-shell condition, if

$$(L_0 - a)|\eta\rangle = 0. \tag{2.1}$$

Here L_0 is normal-ordered and a is a constant that takes into account the freedom in subtracting an infinite constant in the normal ordering process and needs to be fixed later. A state $|\phi\rangle$ is called physical if it satisfies the mass-shell condition and furthermore obeys

$$L_m|\phi\rangle = 0, \quad m > 0. \tag{2.2}$$

Finally, a state $|\psi\rangle$ is called spurious, if it satisfies the mass-shell condition and is orthogonal to all physical states.

1. Show that a state that is spurious and physical has zero norm. Those states are called null states.

Next, we consider the first excited state in open string theory which is given by

$$|\zeta, k\rangle = \zeta \cdot \alpha_{-1}|0, k\rangle \tag{2.3}$$

Here $|0, k\rangle$ is the vacuum, i.e. an un-excited string having momentum p^μ. ζ^μ denotes a polarization vector. The mass of such a state is given by

$$M^2 = \frac{1}{l_s^2}\big(N - a\big), \quad N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n. \tag{2.4}$$

N is called the number operator.

2. Show that the state

$$|\psi\rangle = L_{-1}|0, k\rangle \tag{2.5}$$

is spurious.

3. Consider the cases $a < 1, a = 1, a > 1$ and comment on
   - the mass of $|\zeta, k\rangle$
   - if $|\psi\rangle$ is physical .

Which is the right value for a in order to have a massless vector in the spectrum?

The null states decouple from physical processes as they are orthogonal to all physical states. Therefore one may identify

$$|\phi\rangle \sim |\phi\rangle + |\eta\rangle, \quad |\eta\rangle \quad \text{null.} \tag{2.6}$$

4. Show that the condition

$$L_n|\zeta, k\rangle = 0, \quad n > 0 \tag{2.7}$$

is satisfied automatically for $n > 1$ and leads for $n = 1$ to

$$p^\mu \zeta_\mu |0, k\rangle. \tag{2.8}$$

5. What is the condition

$$p^\mu \zeta_\mu = 0 \tag{2.9}$$

in position space?

6. Show for the null state

$$i\Lambda(k)L_{-1}|0, k\rangle = i\sqrt{2\alpha'}k_\mu \Lambda \alpha^\mu_{-1}|0, k\rangle. \tag{2.10}$$

7. What is therefore the interpretation of the identification (2.6) in position space?

3 Determination of a and D **2 Points**

In the previous exercise we have seen the physical relevance of null states. An important question is therefore under which conditions the theory contains "a lot" of these states. We consider states of the form

$$|\psi\rangle = L_{-1}|\chi_1\rangle, \quad (L_0 - a + 1)|\chi_1\rangle = 0, \quad L_m|\chi_1\rangle = 0, \quad m > 0. \tag{3.1}$$

1. Show that the physical state condition implies in the case $m = 1$

$$L_1|\psi\rangle = 2(a - 1)|\chi_1\rangle = 0. \tag{3.2}$$

Hint: Use the Virasoro algebra.

Next we consider states of the form

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}^2)|\tilde{\chi}\rangle, \quad (L_0 + 1)|\tilde{\chi}\rangle = L_m|\tilde{\chi}\rangle = 0, \quad m > 0. \tag{3.3}$$

2. Show that

$$L_1|\psi\rangle \tag{3.4}$$

leads to $\gamma = \frac{3}{2}$.

3. Show that

$$L_2|\psi\rangle = \big(-13 + \frac{D}{2}\big) \tag{3.5}$$

and conclude that $D = 26$.

3.1 A different way to fix D

Another way to fix D, the number of space-time dimensions, is given by evaluating the normal ordering constant a of L_0 directly. This makes use of the so-called ζ-function regularization. The ζ-function is defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \mathrm{Re}\, s > 1. \tag{3.6}$$

In can be analytically continued to a meromorphic function on the whole complex plane. In particular it obeys

$$\zeta(-1) = -\frac{1}{12}. \tag{3.7}$$

1. Show that a is formally given as

$$a = -\frac{1}{2} \sum_{m=1}^{\infty} [a_m^i, a_{-m}^j] \delta_{ij} \tag{3.8}$$

2. Show that this fixes D by treating the infinite sum as a value of the ζ-function. Hint: So far we have been using the light-cone gauge quantization. Therefore there are just $D-2$ oscillation modes.

4 Lorentz symmetry of the quantum string **3 Points**

On the first sheet you found the currents $j^{\mu\nu}$ of the Lorentz symmetry. The conserved charge

$$J^{\mu\nu} = \int d\sigma j^{\mu\nu} \tag{4.1}$$

becomes a (normal-ordered) operator in the quantum theory. Show that

$$[L_m, J^{\mu\nu}] = 0. \tag{4.2}$$

What does this imply for the spectrum concerning representations of the Lorentz group?