Physikalisches Institut
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Theoretische Physik

Exercise 5
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Exercises String Theory

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1 The conformal and the Lorentz group — 4 points

We recall that the generators of the conformal group can be expressed as follows

translation	$P_\mu = -i\partial_\mu$
dilatation	$D = -ix^\mu \partial_\mu$
rotation	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
special conformal transformation	$K_\mu = -(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

They obey the following algebra

$$\begin{aligned}
[D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu, \quad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}),\\
[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu), \quad [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu),\\
[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})
\end{aligned} \tag{1.1}$$

1. Check the first four relations.
2. Show that the re-defined fields

$$\begin{aligned}
J_{\mu\nu} = L_{\mu\nu} \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu)\\
J_{-1,0} = D \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu)
\end{aligned} \tag{1.2}$$

obey the SO(d+1,1) commutation relations

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}). \tag{1.3}$$

2 Two-point function — 2 points

We consider the two-point function of two primary fields Φ_i, with conformal weight Δ_i, $i = 1, 2$ in 2d CFT

$$G(z_1, z_2) = < \Phi_1(z_1)\Phi_2(z_2) > . \tag{2.1}$$

Recall that under infinitesimal transformation $z \longrightarrow z + \epsilon$ the fields transform as

$$\delta_\epsilon \Phi_i(z) = \big(\epsilon(z)\partial + \Delta_i \partial\epsilon(z)\big)\Phi_i, \quad i = 1, 2. \tag{2.2}$$

1. Show that conformal invariance implies

$$\big(\epsilon(z_1)\partial_1 + \Delta_1 \partial\epsilon(z_1) + \epsilon(z_2)\partial_2 + \Delta_2 \partial\epsilon(z_2)\big)G(z_1, z_2) = 0. \tag{2.3}$$

2. By setting $\epsilon = 1$, show that $G(z_1, z_2)$ is a function of $x = z_1 - z_2$ only.
3. By setting $\epsilon = z$, show that $G(z_1, z_2)$ takes the following form

$$G(x) = \frac{C}{x^{\Delta_1+\Delta_2}}, \tag{2.4}$$

   where C is a constant.
4. By setting $\epsilon = z^2$, show that $G(z_1, z_2)$ vanishes unless $\Delta_1 = \Delta_2$.

3 Global conformal transformations **2 points**

As is known from the lecture, the invariance under global conformal transformations can be encoded in the following differential equations imposed on the correlators of primary fields

$$\begin{aligned} \sum_i \partial_{w_i} < \phi_1(w_1)\ldots\phi_n(w_n) >= 0 \\ \sum_i (w_i\partial_{w_i} + \Delta_i) < \phi_1(w_1)\ldots\phi_n(w_n) >= 0 \\ \sum_i (w_i^2\partial_{w_i} + w_i\Delta_i) < \phi_1(w_1)\ldots\phi_n(w_n) >= 0. \end{aligned} \tag{3.1}$$

Show explicitly for the two- and the three-point function that these relations are indeed valid.

4 Operator product expansion **2 points**

Starting with

$$\delta_{\epsilon,\bar\epsilon}\phi(z,\bar z) = [Q_\epsilon + Q_{\bar\epsilon}, \phi(z,\bar z)], \quad Q_\epsilon = \oint_{C_0} \frac{dz}{2\pi i}\epsilon(z)T(z) \tag{4.1}$$

and

$$\delta_{\epsilon,\bar\epsilon}\phi(z,\bar z) = (\Delta\partial\epsilon + \bar\Delta\bar\partial\epsilon + \epsilon\partial + \bar\epsilon\bar\partial)\phi(z,\bar z), \tag{4.2}$$

Show that

$$T(z)\phi(w,\bar w) = \frac{\Delta}{(z-w)^2}\phi(w,\bar w) + \frac{1}{z-w}\partial_w\phi(w,\bar w) + \text{reg.} \tag{4.3}$$

In the last expression, radial ordering is understood. Hint: Use the operator product expansion

$$\phi_i(z,\bar z)\phi_j(w,\bar w) = \sum_k c_{ijk}(z-w)^{\Delta_k-\Delta_i-\Delta_j}(\bar z-\bar w)^{\bar\Delta_k-\bar\Delta_i-\bar\Delta_j} \tag{4.4}$$

and the Cauchy formula

$$\oint_{C_w} \frac{dz}{2\pi i}\frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!}f^{(n-1)}(w) \tag{4.5}$$

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