Physikalisches Institut
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Exercise 7
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Exercises String Theory

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1 The partition function of the free fermion on the torus **7 points**

We consider the action of a free fermion

$$S = \frac{1}{2\pi} \int d^2 z \big(\bar{\psi} \partial \bar{\psi} + \psi \bar{\partial} \psi \big). \tag{1.1}$$

The fermions may pick up a phase when translated along a period of the torus $\mathbb{C}/\big(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\big)$.

$$\psi(z+\omega_1) = e^{2\pi i v} \psi(z), \qquad \psi(z+\omega_2) = e^{2\pi i u} \psi(z). \tag{1.2}$$

1. Show that the only two values for u and v are 0 and $\frac{1}{2}$ respectively, so that we end up with the four possibilities

$$\begin{aligned} (v,u) &= (0,0) \qquad (R,R) \\ (v,u) &= (0,\tfrac{1}{2}) \qquad (R,NS) \\ (v,u) &= (\tfrac{1}{2},0) \qquad (NS,R) \\ (v,u) &= (\tfrac{1}{2},\tfrac{1}{2}) \qquad (NS,NS) \end{aligned} \tag{1.3}$$

   Hint: Evaluate the equations of motion and take care of the boundary terms. Show that periodic respectively anti-periodic solutions are the only allowed ones. Why are there no anti-periodic boundary conditions in the bosonic case?

The partition function is given for the respective boundary conditions as

$$Z_{v,u} = |d_{v,u}|^2 \tag{1.4}$$

$$\begin{aligned} d_{0,0} &= \frac{1}{\sqrt{2}} \mathrm{Tr}(-1)^F q^{L_0 - \frac{1}{48}}, \\ d_{0,\frac{1}{2}} &= \frac{1}{\sqrt{2}} \mathrm{Tr} q^{L_0 - \frac{1}{48}}, \\ d_{\frac{1}{2},0} &= \mathrm{Tr}(-1)^F q^{L_0 - \frac{1}{48}}, \\ d_{\frac{1}{2},\frac{1}{2}} &= \mathrm{Tr} q^{L_0 - \frac{1}{48}}, \end{aligned}$$

$$NS: \quad L_0 = \sum_{k \in \mathbb{N}_0 + \frac{1}{2}} k b_{-k} b_k, \qquad R: \quad L_0 = \sum_{k \in \mathbb{N}} k b_{-k} b_k + \frac{1}{16}. \tag{1.5}$$

Here F is the fermion number operator

$$F = \sum_{k\ \geq 0} F_k, \qquad F_k = b_{-k} b_k, \quad k > 0. \tag{1.6}$$

F_0 is an operator defined in the space-periodic case, equal to 0, when acting on $|0\rangle$ and 1 when acting on $b_0|0\rangle$.
Furthermore we introduce the Jacobi Theta functions as well as the Dedekind η function as

$$\begin{aligned}
\theta_2(\tau) &= 2q^{1/8} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2, \qquad q = e^{2\pi i \tau}, \\
\theta_3(\tau) &= \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-1/2})^2, \\
\theta_4(\tau) &= \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-1/2})^2, \\
\eta(\tau) &= q^{1/24} \prod_{n=1}^{\infty} (1-q^n).
\end{aligned} \tag{1.7}$$

These transform under modular transformations as

$$\begin{aligned}
\theta_2(\tau+1) = e^{i\pi/4}\theta_2(\tau), &\qquad \theta_2(-1/\tau) = \sqrt{-i\tau}\,\theta_4(\tau), \\
\theta_3(\tau+1) = \theta_4(\tau), &\qquad \theta_3(-1/\tau) = \sqrt{-i\tau}\,\theta_3(\tau), \\
\theta_4(\tau+1) = \theta_3(\tau), &\qquad \theta_4(-1/\tau) = \sqrt{-i\tau}\,\theta_2(\tau), \\
\eta(\tau+1) = e^{i\pi/12}\eta(\tau), &\qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau).
\end{aligned} \tag{1.8}$$

2. Show that

$$\begin{aligned}
d_{0,0} &= 0, \\
d_{0,\frac{1}{2}} &= \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}, \\
d_{\frac{1}{2},0} &= \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}, \\
d_{\frac{1}{2},\frac{1}{2}} &= \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}.
\end{aligned} \tag{1.9}$$

3. Determine the transformation behaviour of the $d_{u,v}$ under the generators of the modular group

$$T : \tau \to \tau + 1, \quad S : \tau \to -\frac{1}{\tau} \tag{1.10}$$

4. Make an ansatz for a modular invariant partition function. What do you conclude concerning the boundary conditions?

2 Fusion rules for the Ising model **3 points**

We consider the minimal model $\mathcal{M}(p,p')$, The conformal families $\phi_{(r,s)}$ can be represented as points in the so-called Kac table, which imposes the following restrictions

$$1 \leq r < p', \quad 1 \leq s < p, \quad \phi_{(r,s)} = \phi_{(p'-r,p-s)}. \tag{2.1}$$

The truncated fusion rules existing between these fields are

$$\begin{gathered}\phi_{(r,s)} \times \phi_{(m,n)} = \sum\nolimits_{\substack{k=1+|r-m| \\ k+r+m=1 mod 2}}^{k_{max}} \sum\nolimits_{\substack{l=1+|s-n| \\ k+s+n=1 mod 2}}^{l_{max}} \phi_{(k,l)}, \\ k_{max} = min\Big(r+m-1, 2p'-1-r-m\Big), \quad l_{max} = min\Big(s+n-1, 2p-1-s-n\Big)\end{gathered} \tag{2.2}$$

This means that the OPE of $\phi_{(r,s)}$ with $\phi_{(m,n)}$ can contain only terms that appear on the right hand side.

1. Work out the truncated fusion rules for $\mathcal{M}(4,3')$.