Exercises String Theory

Prof. Dr. Albrecht Klemm

1 The string compactified on a circle. **10 points**

We consider the bosonic string propagating in 26 dimensional space-time, with X^{25} being a circle of radius R, i.e.

$$X^{25} \sim X^{25} + 2\pi R. \tag{1.1}$$

We recall the expansion of the closed bosonic string

$$X_R^\mu(\sigma^-) = \frac{x^\mu}{2} - \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + \text{oscillators} \tag{1.2}$$

The space-time momentum of the string is given as

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}\left(\alpha_0^\mu + \tilde{\alpha}_0^\mu\right) \tag{1.3}$$

In the following we concentrate on the 25th direction. The periodicity condition (1.1) allows for

$$X^{25}(\sigma) = X^{25}(\sigma + 2\pi) + 2\pi w R. \tag{1.4}$$

and the momentum is quantized as

$$p^{25} = \frac{n}{R}, \qquad n \in \mathbb{N}. \tag{1.5}$$

1. What is the interpretation of w?
2. Show that

$$\begin{aligned} \alpha_0^{25} &= \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} = P_L\sqrt{\frac{\alpha'}{2}} \\ \tilde{\alpha}_0^{25} &= \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} = P_R\sqrt{\frac{\alpha'}{2}}. \end{aligned} \tag{1.6}$$

3. Show that the mass formula respectively the level matching condition are modified to

$$\begin{aligned} \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) &= M^2 \\ nw + N - \tilde{N} &= 0. \end{aligned} \tag{1.7}$$

Hint: Throughout the whole exercise we are neglecting the issues coming from light-cone quantization.

4. What are the massless states in this theory? Show that there are in particular two massless vectors, i.e. we have a $U(1) \times U(1)$ symmetry in the theory. Can you identify the massless states by compactification of the massless spectrum of the closed string in 26 space-time dimensions?

5. Show that the mass-formula is invariant under the exchange

$$n \leftrightarrow w, \qquad R \leftrightarrow R' = \alpha'/R. \tag{1.8}$$

6. Show that there are four additional massless states in the theory at the self-dual radius $R = \sqrt{\alpha'}$.

7. Compute the OPEs of the currents given by

$$\begin{aligned} &\frac{1}{\sqrt{\alpha'}}\partial X^{25}(z), \qquad :\exp\big(\pm 2iX^{25}(z)/\sqrt{\alpha'}\big): \\ &\frac{1}{\sqrt{\alpha'}}\bar{\partial} X^{25}(\bar{z}), \qquad :\exp\big(\pm 2iX^{25}(\bar{z})/\sqrt{\alpha'}\big): \end{aligned} \tag{1.9}$$

Hint: The propagator is normalized as

$$< X^\mu(z) X^\nu(w) >= -\frac{\alpha'}{2}\ln\big(z-w\big) \tag{1.10}$$

You may also want to use results from previous sheets.

8. Show that the modes of the currents form an affine $SU(2) \times SU(2)$ algebra at level 1.

9. The partition function of this theory is given by

$$Z(q,R) = \big(\eta\bar{\eta}\big)^{-1} \sum_{n,w} q^{\frac{\alpha'}{4}P_L^2} \bar{q}^{\frac{\alpha'}{4}P_R^2}, \qquad \eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty}\big(1-q^n\big). \tag{1.11}$$

In addition there is a factor $\big(\eta\bar{\eta}\big)^{-1}$ coming from each additional non-compact direction. By expanding out the partition function up to the first level, show that the mass formula as well as the level matching condition are reproduced.

10. Show that the partition function of the theory at the self-dual radius can be written as

$$Z = |\chi_1(q)|^2 + |\chi_2(q)|^2, \quad \chi_1(q) = \eta^{-1}\sum_n q^{n^2}, \quad \chi_2(q) = \eta^{-1}\sum_n q^{(n+1/2)^2} \tag{1.12}$$

The χ_i are the characters of the affine $su(2)$ algebra. By expanding this expression out find the massless states from above.