Exercises String Theory

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1 The $\mathcal{N}=4$ superconformal algebra — 8 points

We consider the $\mathcal{N}=4$ superconformal algebra, whose field content is given by the energy-momentum tensor $T(z)$, four supercurrents $G^\alpha(z), \bar{G}^\beta(z)$ and three currents $J^a(z)$ forming a current algebra of $SU(2)$ at level k. The algebra is defined by the following OPEs

$$\begin{aligned}
J^a(z)J^b(w) &\sim \frac{k}{2}\frac{\delta^{ab}}{(z-w)^2} + i\epsilon^{abc}\frac{J^c(w)}{(z-w)} \\
J^a(z)G^\alpha(w) &\sim \frac{1}{2}\sigma^a_{\beta\alpha}\frac{G^\beta(w)}{(z-w)}, \qquad J^a(z)\bar{G}^\alpha(w) \sim -\frac{1}{2}\sigma^a_{\beta\alpha}\frac{\bar{G}^\beta(w)}{(z-w)} \\
G^\alpha(z)\bar{G}^\beta(w) &\sim \frac{4k\delta^{\alpha\beta}}{(z-w)^3} + 2\sigma^a_{\beta\alpha}\Big(\frac{2J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}\Big) + 2\delta^{\alpha\beta}\frac{T(w)}{(z-w)}.
\end{aligned} \tag{1.1}$$

The OPEs $G^\alpha(z)G^\beta(w)$ and $\bar{G}^\alpha(z)\bar{G}^\beta(w)$ are regular. The fields enjoy an expansion of the form

$$\begin{aligned}
G^\alpha(z) &= \sum_r G^\alpha_r z^{-r-3/2} \\
J^a(z) &= \sum_n J^a_n z^{-n-1} \\
T(z) &= \sum_m L_m z^{-m-2}.
\end{aligned} \tag{1.2}$$

1. Show that for unitary primary fields one obtains in the Ramond sector $\Delta \geq k/4$, whereas in the NS sector one finds $\Delta - j \geq 0$. Hint: Use the commutation relations derived from the OPEs in order to show the first statement and the spectral flow to derive the second from the first one.

2. Show that the trace of $(-1)^F$ in the Ramond sector obtains contributions from ground states only.

2 The chiral ring — 2 points

In the $\mathcal{N}=2$ algebra, there are chiral fields corresponding to fields with eigenvalues $2\Delta = |q|$. Here we denote by q the eigenvalue of J_0, as usual.

1. Show that the product of two chiral fields $\phi(z), \chi(w)$ has the charge $q_\phi + q_\chi$.

2. Using the operator product expansion

$$\phi(z)\chi(w) = \sum_i (z-w)^{\Delta_{\psi_i}-\Delta_\phi-\Delta_\chi}\psi_i \tag{2.1}$$

show that the only terms contributing to the OPE in the limit $w \to z$ are those stemming from fields ψ_i that are chiral. The chiral fields therefore form a ring. Why not a field?