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Theoretische Physik

Exercise 2
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Exercises String Theory

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1 The Poisson bracket for the classical string **4 Points**

Recall that the closed string mode expansion reads

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma), \tag{1.1}$$

where

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)} \tag{1.2}$$

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}. \tag{1.3}$$

This was derived from the gauge fixed action

$$S = \frac{T}{2}\int d^2\sigma \big(\dot{X}^2 - X'^2\big). \tag{1.4}$$

The canonical momentum conjugated to the variable X^μ is given by

$$P^\mu(\sigma,\tau) = \frac{\delta S}{\delta \dot{X}_\mu} = T\dot{X}^\mu. \tag{1.5}$$

The classical Poisson brackets are given by

$$\Big\{P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\Big\} = 0, \quad \Big\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\Big\} = 0, \quad \Big\{P^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\Big\} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{1.6}$$

Show that this implies the following Poisson brackets for the modes

$$\begin{aligned}\big\{\alpha^\mu_m, \alpha^\nu_n\big\} = \big\{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\big\} = im\eta^{\mu\nu}\delta_{m+n,0}, \quad \big\{\alpha^\mu_m, \tilde{\alpha}^\nu_n\big\} = 0 \\ \big\{x^\mu, x^\nu\big\} = \big\{p^\mu, p^\nu\big\} = 0, \quad \big\{x^\mu, p^\nu\big\} = \eta^{\mu\nu} \end{aligned} \tag{1.7}$$

$$\big\{x^\mu, \tilde{\alpha}^\nu_n\big\} = \big\{p^\mu, \tilde{\alpha}^\nu_n\big\} = 0 = \big\{x^\mu, \alpha^\nu_n\big\} = \big\{p^\mu, \alpha^\nu_n\big\} \tag{1.8}$$

Hint: The Fourierexpansion of the Delta-distribution on the interval $[0,\pi]$ reads

$$\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{2in(\sigma-\sigma')}. \tag{1.9}$$

2 The Virasoro algebra **3 Points**

Recall that the energy momentum tensor

$$T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} \tag{2.1}$$

vanishes in two dimensions. Show that its non-vanishing components in light-cone coordinates

$$\sigma^{\pm} = \tau \pm \sigma \tag{2.2}$$

are given by

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu = 0, \quad T_{--} = \partial_- X^\mu \partial_- X_\mu = 0, \tag{2.3}$$

whereas the other components vanish automatically

$$T_{+-} = T_{-+} = 0. \tag{2.4}$$

Show that the Fourier expansions read

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)}, \quad T_{++} = 2l_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau+\sigma)}, \tag{2.5}$$

where the coefficients are given by

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n. \tag{2.6}$$

Show that the Poisson bracket of two modes is given by

$$\big\{L_n, L_m\big\} = i(n-m)L_{n+m}. \tag{2.7}$$

The Virasoro algebra appears also from another - but not unrelated - point of view. The choice of world-sheet metric

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.8}$$

does not fix the diffeomorphism symmetry completely, but allows for a re-definition of the light-cone coordinates

$$\sigma^+ \to \sigma^{+\prime}(\sigma^+), \quad \sigma^- \to \sigma^{-\prime}(\sigma^-). \tag{2.9}$$

1. Show that the action is invariant[1] under

$$\delta X^\mu = a_n e^{2in\sigma^-} \partial_- X^\mu. \tag{2.10}$$

2. Show that the corresponding current is given by

$$j = T \partial_- X^\mu \partial_- X_\mu e^{2in\sigma^-} \tag{2.11}$$

3. Show that the corresponding charge

$$Q_n = \int d\sigma j^0 \tag{2.12}$$

   is given by L_n.

[1] We just consider a part of the symmetry for simplicity.

3 A first glimpse at the quantization of the bosonics string **3 Points**

3.1 New coordinates

We consider again the following range of worldsheet coordinates

$$\tau \in [-\infty, \infty], \quad \sigma \in [0, \pi]. \tag{3.1}$$

First we do a Wick rotation, sending

$$\tau \to -i\tau. \tag{3.2}$$

By this the metric on the world-sheet becomes Euclidean. Show that by introducing

$$\zeta = 2(\tau - i\sigma), \quad \bar{\zeta} = 2(\tau + i\sigma), \qquad z = e^{\zeta}, \quad \bar{z} = e^{\bar{\zeta}}. \tag{3.3}$$

the expansion (1.2) reads (we just consider one component and therefore omit space-time indices)

$$X(z, \bar{z}) = x - i\frac{l_s^2}{4} p \log(|z|^2) + i\frac{l_s}{2} \sum_{n \neq 0} \frac{1}{n} \Big(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n} \Big) \tag{3.4}$$

3.2 Normal ordering ambiguities

We canonically quantize the string by mapping the Fourier-modes to operators

$$\alpha_n, \tilde{\alpha}_n \to \hat{\alpha}_n, \hat{\tilde{\alpha}}_n, \tag{3.5}$$

such that the Poisson bracket is mapped to the commutator

$$\{\cdot, \cdot\} \to i[\cdot, \cdot]. \tag{3.6}$$

Therefore one has to deal with normal ordering ambiguities. The normal ordering is defined by[2]

$$\begin{aligned} :xp: = :px: = \ xp, \quad :\alpha_m \alpha_{-n}: = :\alpha_{-n}\alpha_m: = \ \alpha_{-n}\alpha_m \\ :\tilde{\alpha}_m \tilde{\alpha}_{-n}: = :\tilde{\alpha}_{-n}\tilde{\alpha}_m: = \ \tilde{\alpha}_{-n}\tilde{\alpha}_m, \quad m, n \in \mathbb{N}_0. \end{aligned} \tag{3.7}$$

Show that

$$X(z, \bar{z})X(w, \bar{w}) = \ :X(z, \bar{z})X(w, \bar{w}): - \frac{l_s^2}{4} \log|z - w|^2. \tag{3.8}$$

[2]In the following we omit the hat for the operator symbol.