Exercises String Theory

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1 The string compactified on a circle — 10 points

We consider the bosonic string propagating in 26 dimensional space-time, with X^{25} being a circle of radius R, i.e.

$$X^{25} \sim X^{25} + 2\pi R. \tag{1.1}$$

We recall the expansion of the closed bosonic string

$$X^\mu_R(\sigma^-) = \frac{x^\mu}{2} - \sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0 + \tilde{\alpha}^\mu_0)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha^\mu_0 - \tilde{\alpha}^\mu_0)\sigma + \text{oscillators} \tag{1.2}$$

The space-time momentum of the string is given as

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}\left(\alpha^\mu_0 + \tilde{\alpha}^\mu_0\right) \tag{1.3}$$

In the following we concentrate on the 25th direction. The periodicity condition (1.1) allows for

$$X^{25}(\sigma) = X^{25}(\sigma + 2\pi) + 2\pi w R. \tag{1.4}$$

and the momentum is quantized as

$$p^{25} = \frac{n}{R}, \qquad n \in \mathbb{N}. \tag{1.5}$$

1. What is the interpretation of w?
2. Show that

$$\begin{aligned} \alpha^{25}_0 &= \left(\frac{n}{R} + \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} = P_L\sqrt{\frac{\alpha'}{2}} \\ \tilde{\alpha}^{25}_0 &= \left(\frac{n}{R} - \frac{wR}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} = P_R\sqrt{\frac{\alpha'}{2}}. \end{aligned} \tag{1.6}$$

3. Show that the mass formula respectively the level matching condition are modified to

$$\begin{aligned} \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) &= M^2 \\ nw + N - \tilde{N} &= 0. \end{aligned} \tag{1.7}$$

   Hint: Throughout the whole exercise we are neglecting the issues coming from light-cone quantization.

4. What are the massless states in this theory? Show that there are in particular two massless vectors, i.e. we have a $U(1) \times U(1)$ symmetry in the theory. Can you identify the massless states by compactification of the massless spectrum of the closed string in 26 space-time dimensions?

5. Show that the mass-formula is invariant under the exchange

$$n \leftrightarrow w, \qquad R \leftrightarrow R' = \alpha'/R. \tag{1.8}$$

6. Show that there are four additional massless states in the theory at the self-dual radius $R = \sqrt{\alpha'}$.

7. Compute the OPEs of the currents given by

$$\begin{aligned} &\frac{1}{\sqrt{\alpha'}}\partial X^{25}(z), \qquad :\exp\big(\pm 2iX^{25}(z)/\sqrt{\alpha'}\big): \\ &\frac{1}{\sqrt{\alpha'}}\bar{\partial} X^{25}(\bar{z}), \qquad :\exp\big(\pm 2iX^{25}(\bar{z})/\sqrt{\alpha'}\big): \end{aligned} \tag{1.9}$$

Hint: The propagator is normalized as

$$< X^\mu(z) X^\nu(w) >= -\frac{\alpha'}{2}\ln\big(z-w\big) \tag{1.10}$$

You may also want to use results from previous sheets.

8. Show that the modes of the currents form an affine $SU(2) \times SU(2)$ algebra at level 1.