Physikalisches Institut
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Exercise Sheet 1
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Exercises in Stringtheory

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`http://www.th.physik.uni-bonn.de/klemm/strings1617/`

–Presence exercises–

1 The relativistic point particle

We consider the action for a relativistic point particle

$$S_{pp} = -m \int ds = -m \int \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} d\tau \,, \tag{1}$$

where τ parameterizes the worldline of the particle and $\eta_{\mu\nu}$ is the D-dimensional Minkowski metric.

1. Show that the action is invariant under Poincaré transformations.
2. Show that the action is invariant under reparametrizations $\tau \to \tau'(\tau)$.
3. Show that
$$p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\eta_{\nu\rho}\dot{X}^\nu \dot{X}^\rho}} \,, \tag{2}$$
is a conserved quantity. Do this once by evaluating the Euler Lagrange equations and once by exploiting the symmetry $X^\mu \to X^\mu + b^\mu$. Hint: For the latter, assume that b^μ depends on τ to do a partial integration and only then restrict to constant b^μ.
4. Why is this action inappropriate to describe massless particles?
5. Show that
$$S_e = -\frac{1}{2} \int d\tau e \left(-\frac{1}{e^2} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} + m^2 \right) \,, \tag{3}$$
is equivalent to the action (1). Hint: Integrate out e.
6. Explain the statement "We have coupled the particle to worldline gravity". What kind of field is e?
7. Show the invariance of the new action (11) under reparametrizations of τ. How does e transform?

2 The Nambu-Goto action versus the Polyakov action

The Nambu-Goto action for a string is given by

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)}\,. \tag{4}$$

Here σ^α, $\alpha = 0, 1$ label the worldsheet time τ and space σ.

1. Write down explicitly the action (17), i.e. without referring to σ^α, but to τ and σ instead.
2. What is the geometric interpretation of this action?
3. Show that the Polyakov action

$$S_P = -\frac{T}{2} \int d^2\sigma \sqrt{-h}\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)\,, \tag{5}$$

is equivalent to the Nambu-Goto action.

–Homework–

3 Symmetries of the string and their implications 5 Points

3.1 Global symmetries 2 Points

Show that the Polyakov action is invariant under Poincaré transformations

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + b^\mu\,. \tag{6}$$

Evaluate the corresponding conserved currents using the Noether procedure which we briefly recall. If the Lagrangian is invariant under an infinitesimal transformation of the fields given by

$$\phi^a \to \phi^a + \delta\phi^a\,, \quad \delta\phi^a = \epsilon^i h^a_i(\phi^b)\,, \tag{7}$$

where ϵ^i is infinitesimal and h^a_i denotes a function of the fields ϕ^a, then the current j^α_i defined by

$$\epsilon^i j^\alpha_i = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a\,, \tag{8}$$

is conserved. Note that i might be a multi-index. The infinitesimal variations for Poincaré transformations are respectively given by

$$X^\mu \to X^\mu + \epsilon^\mu\,, \quad X_\mu \to X_\mu \epsilon a_{\mu\nu} X^\nu\,, \quad a_{\mu\nu} = -a_{\nu\mu}\,. \tag{9}$$

Evaluate the currents using

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{10}$$

See also the exercise (3.2) and equation (46).

3.2 Local symmetries

1. Show that the action is invariant under worldsheet reparametrizations

$$\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta) \,. \tag{11}$$

2. Show that the action is invariant under Weyl transformations

$$h_{\alpha\beta} \to e^{\phi(\sigma^\alpha)} h_{\alpha\beta} \,. \tag{12}$$

3. It can be shown (see. e.g. *Martin Schottenloher - A mathematical introduction to conformal field theory*) that locally there exists parametrizations such that the metric is of the form

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{13}$$

Why is it in general not possible to choose this metric globally?

4 Two-dimensional gravity

Show that the energy momentum tensor in two dimension vanishes identically due to Einstein's equation. Hint: The only non-vanishing component of the Riemann tensor in two dimensions is

$$R_{0101} = -R_{0110} = -R_{1001} = R_{1010} \,. \tag{14}$$

5 The equations of motion

In the following, we consider the range of worldsheet coordinates given by

$$\tau \in \mathbb{R} \,, \quad \sigma \in [0, \pi] \,. \tag{15}$$

1. Show that for the gauge fixed metric (39) the action takes the simple form

$$S = \frac{T}{2} \int d^2\sigma \left(\dot{X}^2 - X'^2 \right) \,. \tag{16}$$

Here we have denoted by $\dot{X}$ and X' the derivatives with respect to τ and σ.

2. Derive the equations of motion. In addition, show that there is a boundary term

$$-T \int \tau' \left(X'_\mu \delta X^\mu \Big|_{\sigma=\pi} - X'_\mu \delta X^\mu \Big|_{\sigma=0} \right) \,. \tag{17}$$

3. Show that there are three possibilities in order to make the boundary term vanish:

a)

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau) \tag{18}$$

b)

$$X'_{\mu}(\sigma,\tau) = 0, \quad \sigma = 0,\pi \tag{19}$$

c)

$$X^{\mu}\Big|_{\sigma=0} = X_0^{\mu}, \quad X^{\mu}\Big|_{\sigma=\pi} = X_{\pi}^{\mu} \tag{20}$$

Comment on the physical interpretations of the three boundary conditions. Why is the last one "strange"?

The general solution to the equations of motion takes the form

$$X^{\mu}(\sigma,\tau) = f(\sigma)g(\tau)\,. \tag{21}$$

4. Show that for closed strings

$$\frac{\partial^2 f(\sigma)}{\partial\sigma^2} = cf(\sigma)\,, \quad \frac{\partial^2 g(\tau)}{\partial\tau^2} = cg(\tau), \quad c = -4m^2, \quad m \in \mathbb{Z}\,. \tag{22}$$

5. Conclude that the general solution for closed strings takes the form

$$X^{\mu}(\sigma,\tau) = X_R^{\mu}(\tau-\sigma) + X_L^{\mu}(\tau+\sigma)\,, \tag{23}$$

where

$$\begin{aligned} X_R^{\mu} &= \frac{1}{2}x^{\mu} + \frac{1}{2}l_s^2 p^{\mu}(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^{\mu} e^{-2in(\tau-\sigma)}\,, \\ X_L^{\mu} &= \frac{1}{2}x^{\mu} + \frac{1}{2}l_s^2 p^{\mu}(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^{\mu} e^{-2in(\tau+\sigma)}\,. \end{aligned} \tag{24}$$

Here n runs over all non-zero integers. Which conditions have to be imposed on α_n^{μ}, $\tilde{\alpha}_n^{\mu}$ in order to make X^{μ} real?