Physikalisches Institut
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Theoretische Physik

Exercise Sheet 2
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Exercises in Stringtheory

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`http://www.th.physik.uni-bonn.de/klemm/strings1617/`

1 The Poisson bracket for the classical string (4 Points)

Recall that the closed string mode expansion reads

$$X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)\,, \tag{1}$$

where

$$\begin{aligned} X_R^\mu =& \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}\,,\\ X_L^\mu =& \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}\,. \end{aligned} \tag{2}$$

This was derived from the gauge fixed action

$$S = \frac{T}{2}\int d^2\sigma(\dot{X}^2 - X'^2)\,. \tag{3}$$

The canonical momentum conjugated to the variable X^μ is given by

$$P^\mu(\sigma,\tau) = \frac{\delta S}{\delta \dot{X}_\mu} = T\dot{X}^\mu\,. \tag{4}$$

The classical Poisson brackets are given by

$$\{P^\mu(\sigma,\tau),P^\nu(\sigma',\tau)\} = 0\,,\quad \{X^\mu(\sigma,\tau),X^\nu(\sigma',\tau)\} = 0\,,\quad \{P^\mu(\sigma,\tau),X^\nu(\sigma',\tau)\} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{5}$$

Show that this implies the following Poisson brackets for the modes:

$$\begin{gathered} \{\alpha_m^\mu,\alpha_n^\nu\} = \{\tilde{\alpha}_m^\mu,\tilde{\alpha}_n^\nu\} = im\eta^{\mu\nu}\delta_{m+n,0}\\ \{\alpha_m^\mu,\tilde{\alpha}_n^\nu\} = \{x^\mu,x^\nu\} = \{p^\mu,p^\nu\} = 0,\quad \{x^\mu,p^\nu\} = \eta^{\mu\nu}\\ \{x^\mu,\tilde{\alpha}_n^\nu\} = \{p^\mu,\tilde{\alpha}_n^\nu\} = \{x^\mu,\alpha_n^\nu\} = \{p^\mu,\alpha_n^\nu\} = 0 \end{gathered} \tag{6}$$

Hint: The Fourier expansion of the Delta distribution on the interval $[0,\pi]$ reads

$$\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{2in(\sigma-\sigma')}\,. \tag{7}$$

2 The Virasoro algebra (3 Points)

Recall that the energy momentum tensor

$$T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}\,, \tag{8}$$

vanishes in two dimensions. Show that its components in light-cone coordinates

$$\sigma^{\pm} = \tau \pm \sigma\,, \tag{9}$$

are given by

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu\,, \quad T_{--} = \partial_- X^\mu \partial_- X_\mu\,, \tag{10}$$

whereas the other components vanish automatically, i.e.

$$T_{+-} = T_{-+} = 0\,. \tag{11}$$

Show that the Fourier expansions read

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)}\,, \quad T_{++} = 2l_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau+\sigma)}\,, \tag{12}$$

where the coefficients are given by

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n\,, \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n\,. \tag{13}$$

Show that the Poisson bracket of two modes is given by

$$\{L_n, L_m\} = i(n-m)L_{n+m}\,. \tag{14}$$

The Virasoro algebra appears also from another - but no unrelated - point of view. The choice of world-sheet metric

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{15}$$

does not fix the diffeomorphism symmetry completely but allows for a redefinition of the light-cone coordinates

$$\sigma^+ \to \sigma^{+'}(\sigma^+)\,, \quad \sigma^- \to \sigma^{-'}(\sigma^-)\,. \tag{16}$$

1. Show that the action is invariant[1] under

$$\delta X^\mu = a_n e^{2in\sigma^-} \partial_- X^\mu\,. \tag{17}$$

2. Show that the corresponding current is given by

$$j = T \partial_- X^\mu \partial_- X_\mu e^{2in\sigma^-}\,. \tag{18}$$

3. Show that the corresponding charge

$$Q_n = \int d\sigma j^0\,, \tag{19}$$

   is given by L_n.

[1] We just consider a part of the symmetry for simplicity.

3 A first glimpse at the quantization of the bosonic string (3 Points)

3.1 New coordinates

We consider again the following range of worldsheet coordinates

$$\tau \in [-\infty, \infty]\,, \quad \sigma \in [0, \pi]\,. \tag{20}$$

First we do a Wick rotation, sending

$$\tau \to -i\tau\,. \tag{21}$$

By this the metric on the worldsheet becomes Euclidean. Show that by introducing

$$\zeta = 2(\tau - i\sigma)\,, \quad \bar{\zeta} = 2(\tau + i\sigma)\,, \quad z = e^{\zeta}\,, \quad \bar{z} = e^{\bar{\zeta}}\,, \tag{22}$$

the expansion (2) reads (we just consider one component and therefore omit the spacetime indices)

$$X(z, \bar{z}) = x - i\frac{l_s^2}{4} p \log(|z|^2) + i\frac{l_s}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n} \right)\,. \tag{23}$$

3.2 Normal ordering ambiguities

We canonically quantize the string by mapping the Fourier modes to operators

$$\alpha_n, \tilde{\alpha}_n \to \hat{\alpha}_n, \hat{\tilde{\alpha}}_n\,, \tag{24}$$

such that the Poisson bracket is mapped to the commutator

$$\{\cdot, \cdot\} \to i\,[\cdot, \cdot]\,. \tag{25}$$

Therefore one has to deal with normal ordering ambiguities. The normal ordering is defined by[2]

$$\begin{aligned} :xp: \,=\, :px: \,=\, xp\,, \quad :\alpha_m \alpha_{-n}: \,=\, :\alpha_{-n}\alpha_m: \,=\, \alpha_{-n}\alpha_m\,, \\ :\tilde{\alpha}_m \tilde{\alpha}_{-n}: \,=\, :\tilde{\alpha}_{-n}\tilde{\alpha}_m: \,=\, \tilde{\alpha}_{-n}\tilde{\alpha}_m\,, \quad m, n \in \mathbb{N}_0\,. \end{aligned} \tag{26}$$

Show that

$$X(z, \bar{z}) X(w, \bar{w}) = \,:X(z, \bar{z}) X(w, \bar{w}): - \frac{l_s^2}{4} \log|z - w|^2\,. \tag{27}$$

[2] In the following we omit the hat for the operators.