Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 7
1.11.2016
WS 16/17

Exercises in Stringtheory

Prof. Dr. Albrecht Klemm
Sheets: Thorsten Schimannek, Tutorials: Urmi Ninad

Hand in: 8.12.2016

`http://www.th.physik.uni-bonn.de/klemm/strings1617/`

1 The Virasoro algebra from the energy-momentum tensor (4 points)

As you know, the OPE of the chiral energy-momentum tensor in a two-dimensional conformal field theory is given by

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots\,, \tag{1}$$

where regular terms have been omitted as usual. Moreover, the mode expansion of $T(z)$ reads

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\,, \quad \text{where} \quad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z)\,. \tag{2}$$

Use this to show that the modes generate the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,. \tag{3}$$

Hint: Remember how the contour integral of a commutator has to be interpreted in the light of the radial ordering prescription. *The interplay between commutators, contour integrals and radial ordering is at first a little subtle but - at least from a practical point of view - important. Do yourself a favour and make sure that you know exactly what is going on.*

2 Free fermions (6 points)

Consider the action of a free Majorana fermion in two-dimensional Minkowski space with metric $h_{\alpha\beta} = \mathrm{diag}(+1, -1)$,

$$S_M = \frac{1}{4\pi\kappa}\int dx^0 dx^1 \sqrt{|h|}(-1)\bar{\Psi}\gamma^\alpha\partial_\alpha\Psi\,, \tag{4}$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$. The two-by-two matrices $\{\gamma^\alpha\}$ satisfy the Clifford algebra

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2h^{\alpha\beta}\mathbb{I}_2\,, \tag{5}$$

and a choice of basis is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6}$$

1. Write down the Majorana condition in two dimensions and show that it implies that the components of Ψ

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} , \tag{7}$$

are real. *Note: Strictly speaking there are two different but equivalent Majorana conditions that one can impose. The other choice would make ψ purely imaginary.*

We now perform a Wick rotation sending $x_1 \to ix_1$ and $\partial_1 \to -i\partial_1$.

2. Why do we have to change the γ-matrices and a possible choice of basis is now given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ? \tag{8}$$

Write down the Wick rotated action S_E.

3. Change coordinates via $z = x^0 + ix^1$ and its complex conjugate and show that in these coordinates the action reads

$$S_E = \frac{1}{4\pi\kappa} \int dz d\bar{z} \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right) . \tag{9}$$

4. Derive the equations of motion for the (real) fields ψ and $\bar{\psi}$.

5. Derive the propagator

$$\langle \psi(z) \psi(w) \rangle = \frac{\kappa}{z - w} . \tag{10}$$

To this end use the relation $\delta(z) = \bar{\partial} z^{-1}$, which is of course to be understood in a distributional sense. Its validity can be shown using the Cauchy-Pompeiu formula.

6. Show that the action is invariant under conformal transformations if the conformal dimensions of ψ and $\bar{\psi}$ are $(h, \bar{h}) = (1/2, 0)$ and $(h, \bar{h}) = (0, 1/2)$ respectively.

Due to the fermionic nature of the fields, there are two different possibilities for their behaviour under rotations by 2π. Focussing on the chiral sector, on the complex plane these are given by

$$\begin{aligned} \psi(e^{2\pi i} z) &= +\psi(z) \qquad \text{Neveu-Schwarz sector (NS)} , \\ \psi(e^{2\pi i} z) &= -\psi(z) \qquad \text{Ramond sector (R)} . \end{aligned} \tag{11}$$

Using the information about the conformal weight, we can then write down the mode expansion

$$\psi(z) = \sum_r \psi_r z^{-r-\frac{1}{2}} , \tag{12}$$

with $r \in \mathbb{Z} + 1/2$ for the Neveu-Schwarz sector and $r \in \mathbb{Z}$ for the Ramond sector. Furthermore, again due to the fermionic nature, radial ordering now works as

$$R\left(\Psi(z)\Theta(w)\right) := \begin{cases} +\Psi(z)\Theta(w) & \text{for} \quad |z| > |w| \\ -\Theta(w)\Psi(z) & \text{for} \quad |w| > |z| \end{cases} . \tag{13}$$

1. Show that
$$\{\psi_r, \psi_s\} = \kappa \delta_{r+s,0} \,. \tag{14}$$

2. Calculate the *canonical* energy-momentum tensor which for a theory with fields ϕ_i reads
$$T^c_{\mu\nu} = 8\pi\kappa\gamma \left(-\eta_{\mu\nu}\mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial \left(\partial^\mu \phi_i\right)} \partial_\nu \phi_i \right) \,, \tag{15}$$
where γ is a normalisation constant. Use the equations of motion to show that
$$T_{zz} = \gamma \psi \partial \psi \,, \quad T_{z\bar{z}} = T_{\bar{z}z} = 0 \,, \quad T_{\bar{z}\bar{z}} = \gamma \bar{\psi} \bar{\partial} \bar{\psi} \,. \tag{16}$$
In the quantum theory these expression have to be interpreted with normal ordering, e.g. $T(z) = T_{zz} = \gamma N(\psi\partial\psi)$.

3. Show that for fermionic fields
$$N(\psi\theta)_r = -\sum_{s>-h^\theta} \psi_{r-s}\theta_s + \sum_{s\leq -h^\phi} \theta_s \psi_{r-s} \,, \tag{17}$$
where h^ϕ is the conformal weight of the field ϕ and use this expression to show that the Laurent modes of the energy-momentum tensor are given by
$$L_m = \gamma \sum_{s>-\frac{3}{2}} \psi_{m-s}\psi_s \left(s + \frac{1}{2}\right) - \gamma \sum_{s\leq -\frac{3}{2}} \psi_s \psi_{m-s} \left(s + \frac{1}{2}\right) \,. \tag{18}$$

4. Calculate
$$[L_m, \psi_r] = \gamma\kappa(-m - 2r)\psi_{m+r} \,, \tag{19}$$
and use this to argue that $\gamma\kappa = 1/2$. Usually one also chooses the normalisation $\kappa = 1$.

5. Finally, use the Virasoro algebra to derive the expression
$$\langle 0 | L_2 L_{-2} | 0 \rangle = \frac{c}{2} \,, \tag{20}$$
and from this conclude that the central charge for a conformal field theory given by a real free fermion is $c = 1/2$.