Physikalisches Institut
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Exercise Sheet 9
19.1.2017
WS 16/17

Exercises in Stringtheory

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Hand in: 26.1.2017

`http://www.th.physik.uni-bonn.de/klemm/strings1617/`

1 Poisson resummation formula (5 pts.)

The Poisson resummation formula is an important tool to determine modular properties of special functions. In this exercise we use the Poisson resummation technique to determine the modular properties of the Dedekind eta function $\eta(\tau)$.

1. Given a function $f(x)$ which approaches zero suitably fast for $x \to \pm\infty$, i.e. to be precise we consider $f(x)$ to be a Schwartz function. Then the Fourier transformed function $\hat{f}(p)$ is given by

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x p} dx \,. \tag{1}$$

   Prove the Poisson resummation formula for a Schwartz function f

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \,. \tag{2}$$

   Hint: What are the Fourier coefficients of the periodic function $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$?

2. Use the Poisson resummation formula to derive a summation identity for the function

$$f(x) = e^{-ax^2+bx} \,, \quad \mathrm{Re}(a) > 0 \,. \tag{3}$$

3. Use the Jacobi triple product identity

$$\prod_{n=1}^{\infty} (1-q^n)(1+q^{n-1/2}y)(1+q^{n-1/2}y^{-1}) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} y^n \,, \quad |q|<1 \; y \neq 0 \,, \tag{4}$$

   to derive the infinite sum formula

$$\eta(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3}{2}(n-\frac{1}{6})^2} \,, \tag{5}$$

   for the Dedekind eta function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \,, \quad q = e^{2\pi i \tau} \,. \tag{6}$$

4. Apply the result of 2. to determine the behavior of the Dedekind eta function $\eta(\tau)$ with respect to modular transformations. *Hint: In order to find the modular formula* $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$*, split the infinite sum into three suitable infinite sums after Poisson resummation.*

2 Bosonic closed strings on the circle (10 pts.)

We consider the bosonic string propagating in 26 dimensional space-time, with X^{25} being a circle of radius R, i.e.

$$X^{25} \sim X^{25} + 2\pi R \,. \tag{7}$$

We recall the expansion of the closed bosonic string

$$X^\mu = \frac{1}{2}(x_0^\mu + \tilde{x}_0^\mu) - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in\sigma^-} + \tilde{\alpha}_n^\mu e^{-in\sigma^+}\right) \,. \tag{8}$$

The space-time momentum of the string is given as

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu) \,, \tag{9}$$

and the momentum is quantized as

$$p^{25} = \frac{n}{R} \,, \quad n \in \mathbb{Z} \,. \tag{10}$$

Moreover, the periodicity condition allows for twisted boundary conditions

$$X^{25}(\sigma) = X^{25}(\sigma + 2\pi) + 2\pi w R \,. \tag{11}$$

1. What is the interpretation of w?

2. Show that

$$\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} + \frac{wR}{\alpha'}\right) = \sqrt{\frac{\alpha'}{2}}P_L \,, \quad \tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} - \frac{wR}{\alpha'}\right) = \sqrt{\frac{\alpha'}{2}}P_R \,. \tag{12}$$

3. Show that the mass formula and the level matching condition are modified to

$$\frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) = M^2 \,, nw + N - \tilde{N} = 0 \,. \tag{13}$$

   Hint: Throughout the while exercise we are neglecting the issues coming from light-cone quantization.

4. Determine the massless spectrum and explain how it is obtained from the uncompactified 26-dimensional theory by Kaluza-Klein reduction.

5. Show that the mass-formula is invariant under the exchange

$$n \leftrightarrow w \,, \quad R \leftrightarrow R' = \alpha'/R \,. \tag{14}$$

6. Show that there are four additional massless states in the theory at the self-dual radius $R = \sqrt{\alpha'}$.

7. Compute the OPEs of the currents given by

$$\begin{aligned} &\frac{1}{\sqrt{\alpha'}}\partial X^{25}(z)\,, \quad &&: \exp\left(\pm 2iX^{25}(z)/\sqrt{\alpha'}\right) :\,, \\ &\frac{1}{\sqrt{\alpha'}}\bar{\partial} X^{25}(\bar{z})\,, \quad &&: \exp\left(\pm 2iX^{25}(\bar{z})/\sqrt{\alpha'}\right) :\,. \end{aligned} \tag{15}$$

Hint: The propagator is normalized as

$$\langle X^{\mu}(z)X^{\nu}(w)\rangle = -\frac{\alpha'}{2}\ln(z-w)\,. \tag{16}$$

You may also want to use results from the previous sheets.

8. Show that the modes of the currents form an affine $SU(2)\times SU(2)$ algebra at level 1.

9. The partition function of this theory is given by

$$Z(q,R) = (\eta\bar{\eta})^{-1}\sum_{n,w} q^{\frac{\alpha'}{4}P_L^2}\bar{q}^{\frac{\alpha'}{4}P_R^2}\,, \quad \eta = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{17}$$

In addition there is a factor $(\eta\bar{\eta})^{-1}$ coming from each additional non-compact direction. By expanding out the partition function up to the first level, show that the mass formula as well as the level matching condition are reproduced.