Exercises in Superstring Theory

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1 Superstring theory SCFT

Now we turn back to the superstring theory action given by the supersymmetry extension of the Polyakov action. This is a 2d $N = 1$ supersymmetry with scalar multiplet (X^μ, ψ^μ, F^μ) and supergravity multiplet $(e^a_\alpha, \chi_\alpha, A)$. The complete action is invariant under local supersymmetry transformations. Similarly as in the bosonic string, we can make use of the worldsheet symmetries to gauge away the degrees of freedom given by the worldsheet metric and the worldsheet gravitino, i.e. we fix a **superconformal gauge** locally. At the end of the day our action reduces to

$$S_P|_{SUGRA} \xrightarrow{\text{sup. conf. gauge}} S_P|_{SCFT} = \tfrac{1}{4\pi}\int d^2z\Big(\tfrac{2}{\alpha'}\partial X^\mu\bar\partial X_\mu + \psi^\mu\bar\partial\psi_\mu + \bar\psi^\mu\partial\bar\psi_\mu\Big)\,. \tag{1.1}$$

For notational convenience we set $\alpha' = 2$ and stick with dimensionless fields. For the action given in (1.1), the worldsheet energy-momentum tensor T reads

$$T(z) = -\frac{1}{2} : \partial X^\mu(z)\partial X_\mu(z) : -\frac{1}{2} : \psi^\mu(z)\partial\psi_\mu(z) : \,. \tag{1.2}$$

Similar expressions are given for $\bar T(\bar z)$. Moreover, the worldsheet supercurrents read

$$T_F(z) = \frac{i}{2} : \psi^\mu(z)\partial X_\mu(z) :\,, \quad \bar T_F(y\bar z) = \frac{i}{2} : \bar\psi^\mu(\bar z)\bar\partial X_\mu(\bar z) : \,. \tag{1.3}$$

You might recall from previous exercises the following OPEs[1]:

$$X^\mu(z)X^\nu(w) \sim -\eta^{\mu\nu}\ln(z-w)\,, \quad \bar X^\mu(\bar z)\bar X^\nu(\bar w) \sim -\eta^{\mu\nu}\ln(\bar z-\bar w)\,, \tag{1.4}$$

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}\,, \qquad \bar\psi^\mu(\bar z)\bar\psi^\nu(\bar w) \sim \frac{\eta^{\mu\nu}}{\bar z-\bar w}\,. \tag{1.5}$$

The rest of the OPEs among X^μ and ψ^ν are trivial.

1. Show that $\psi^\mu(z)$ and $T_F(z)$ are primary fields of conformal weight $(h, \bar h)$: $(\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$ respectively.

The superconformal transformations $\delta_\epsilon\phi(z) = -[T_{F_\epsilon}, \phi(z)]$ can be obtained from the generators given by

$$T_{F_\epsilon} = \oint \frac{dz}{2\pi i}\epsilon(z)T_F(z)\,, \tag{1.6}$$

where $\epsilon(z)$ is an anticommutating infinitesimal parameter.

2. Show that (1.3) generate the superconformal transformations

$$\delta_\epsilon X^\mu(z) = \frac{i}{2}\epsilon(z)\psi^\mu(z)\,, \quad \delta_\epsilon\psi^\mu(z) = -\frac{i}{2}\epsilon(z)\partial X^\mu(z)\,. \tag{1.7}$$

[1] Here $X^\mu(z,\bar z) = X^\mu(z) + \bar X^\mu(\bar z)$.

3. Show that the commutator of two superconformal transformations is a conformal transformation, i.e.

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_\xi \,, \quad \xi = \frac{1}{2}\epsilon_1\epsilon_2 \,. \tag{1.8}$$

Similarly, the commutator of a conformal transformation and a superconformal transformation is a superconformal transformation. The conformal transformations thus close to form the **superconformal algebra**. The $N = 1$ superconformal algebra in OPE form is given by

$$T(z)T(w) \sim \frac{\frac{\mathrm{c}}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \,, \tag{1.9}$$

$$T(z)T_F(w) \sim \frac{\frac{3}{2}T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} \,, \tag{1.10}$$

$$T_F(z)T_F(w) \sim \frac{\frac{2\mathrm{c}}{3}}{(z-w)^3} + \frac{2T(z)}{z-w} \,. \tag{1.11}$$

Here $N = 1$ refers to the number of $(\frac{3}{2}, 0)$ currents. In the present case there is also an antiholomorphic copy of the same algebra, so we have an $(N, \bar{N}) = (1, 1)$ **superconformal field theory** (**SCFT**).

4. Using the explicit form of the supercurrent and the energy momentum tensor in (1.3) and (1.2), verify the OPEs (16), (17) and (18). You should find out that $\mathrm{c} = \frac{3}{2}D$.

Denote for the following discussion w as the cylindrical coordinate $w = \sigma^1 + i\sigma^2$. For the closed string $w \sim w + 2\pi$. Lorentz invariance allows two possible periodicity conditions for $\psi^\mu(\bar{\psi}^\mu)$

- Ramond (**R**): $\psi^\mu(w + 2\pi) = +\psi^\mu \,,$
- Neveu-Schwarz (**NS**): $\psi^\mu(w + 2\pi) = -\psi^\mu \,.$

We can rewrite this as $\psi^\mu(w+2\pi) = e^{i2\pi a}\psi^\mu(w)$, where $a \in \{0, \frac{1}{2}\}$. Going back to the conformal plane, the Laurent expansions of the fermionic fields are given by

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+a} \frac{\psi^\mu_r}{z^{r+\frac{1}{2}}} \,, \quad \bar{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+a} \frac{\bar{\psi}^\mu_r}{\bar{z}^{r+\frac{1}{2}}} \,. \tag{1.12}$$

5. Show that $\{\psi^\mu_r, \psi^\nu_s\} = \{\bar{\psi}^\mu_r, \bar{\psi}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}$.

For T and T_F the Laurent expansions are

$$T_F(z) = \sum_{r\in\mathbb{Z}+a} \frac{G_r}{z^{r+\frac{3}{2}}} \,, \quad T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}} \,. \tag{1.13}$$

6. Use the CFT countour calculations to obtain the superconformal algebra commutation relations in terms of the modes, which is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathrm{c}}{12}(m^3 - m)\delta_{m+n,0} \,, \tag{1.14}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{\mathrm{c}}{12}(4r^2 - 1)\delta_{r+s,0} \tag{1.15}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r} \,. \tag{1.16}$$

Mathematically the relations (21), (22) and (23) determine an infinite-dimensional super Lie algebra, A finite algebra contained in the infinite-dimensional one is generated by $\{L_0, L_{\pm 1}, G_{\pm\frac{1}{2}}\}$ and turns out to be $\mathfrak{osp}(1|2)$. The corresponding super group $OSP(1|2)$ plays the same role for SCFTs as $SL(2,\mathbb{C})/\mathbb{Z}_2$ does for the usual CFTs.