Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise Sheet 2
16.10.2018
WS 18/19

Exercises in Superstring Theory

Prof. Dr. Albrecht Klemm
Sheets & Organiztion: César Fierro-Cota

Hand in: 23.10.2018

`http://www.th.physik.uni-bonn.de/klemm/strings1819/`

–Homework–

1 Symmetries of the string and their implications 9 Points

1.1 Global symmetries 3 Points

Show that the Polyakov action is invariant under Poincaré transformations

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + b^\mu \,. \tag{1}$$

Evaluate the corresponding conserved currents using the Noether procedure which we briefly recall. If the Lagrangian is invariant under an infinitesimal transformation of the fields given by

$$\phi^a \to \phi^a + \delta\phi^a \,, \quad \delta\phi^a = \epsilon^i f_i^a(\phi^b) \,, \tag{2}$$

where ϵ^i is infinitesimal and f_i^a denotes a function of the fields ϕ^a, then the current j_i^α defined by

$$\epsilon^i j_i^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \delta\phi^a \,, \tag{3}$$

is conserved. Note that i might be a multi-index. The infinitesimal variations for Poincaré transformations are respectively given by

$$X^\mu \to X^\mu + \epsilon^\mu \,, \quad X_\mu \to X_\mu \epsilon a_{\mu\nu} X^\nu \,, \quad a_{\mu\nu} = -a_{\nu\mu} \,. \tag{4}$$

Evaluate the currents using

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{5}$$

See also the exercise (1.2) and equation (11).

1.2 Local symmetries 6 Points

1. Show that the action is invariant under worldsheet reparametrizations
$$\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta) \,. \tag{6}$$
2. Show that the action is invariant under Weyl transformations
$$h_{\alpha\beta} \to e^{\phi(\sigma^\alpha)} h_{\alpha\beta} \,. \tag{7}$$
3. It can be shown (see. e.g. *Martin Schottenloher - A mathematical introduction to conformal field theory*) that locally there exists parametrizations such that the metric is of the form
$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,. \tag{8}$$
Why is it in general not possible to choose this metric globally?

2 Two-dimensional gravity 4 Points

Show that the energy momentum tensor in two dimension vanishes identically due to Einstein's equation. Hint: The only non-vanishing component of the Riemann tensor in two dimensions is

$$R_{0101} = -R_{0110} = -R_{1001} = R_{1010}\,. \tag{9}$$

3 The equations of motion 11 Points

In the following, we consider the range of worldsheet coordinates given by

$$\tau \in \mathbb{R}\,, \quad \sigma \in [0, \pi]\,. \tag{10}$$

1. Show that for the gauge fixed metric (8) the action takes the simple form

$$S = \frac{T}{2}\int d^2\sigma \left(\dot{X}^2 - X'^2\right)\,. \tag{11}$$

Here we have denoted by $\dot{X}$ and X' the derivatives with respect to τ and σ.

2. Derive the equations of motion. In addition, show that there is a boundary term

$$-T\int \tau' \left(X'_\mu \delta X^\mu \Big|_{\sigma=\pi} - X'_\mu \delta X^\mu \Big|_{\sigma=0} \right)\,. \tag{12}$$

3. Show that there are three possibilities in order to make the boundary term vanish:
   a) $X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau)\,.$
   b) $X'_\mu(\sigma,\tau) = 0, \quad \sigma = 0, \pi\,.$
   c) $X^\mu\Big|_{\sigma=0} = X_0^\mu, \quad X^\mu\Big|_{\sigma=\pi} = X_\pi^\mu\,.$

   Comment on the physical interpretations of the three boundary conditions. Why is the last one "strange"?

The general solution to the equations of motion takes the form

$$X^\mu(\sigma,\tau) = f(\sigma)g(\tau)\,. \tag{13}$$

4. Show that for closed strings

$$\frac{\partial^2 f(\sigma)}{\partial\sigma^2} = cf(\sigma)\,, \quad \frac{\partial^2 g(\tau)}{\partial\tau^2} = cg(\tau), \quad c = -4m^2, \quad m \in \mathbb{Z}\,. \tag{14}$$

5. Conclude that the general solution for closed strings takes the form

$$X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)\,, \tag{15}$$

where

$$\begin{aligned} X_R^\mu =& \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}\,, \\ X_L^\mu =& \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}\,. \end{aligned} \tag{16}$$

Here n runs over all non-zero integers. Which conditions have to be imposed on α_n^μ, $\tilde{\alpha}_n^\mu$ in order to make X^μ real?