Physikalisches Institut
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Theoretische Physik

Exercise Sheet 3
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Exercises in Superstring Theory

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http://www.th.physik.uni-bonn.de/klemm/strings1819/

–Homework–

1 Spacetime conserved quantities & Poisson brackets (12 Points)

1.1 Momentum & Poisson brackets (6 Points)

In this exercise we consider the classical closed string. Recall that its mode expansion reads

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)\,, \tag{1}$$

where[1]

$$\begin{aligned} X^\mu_R =& \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)}\,, \\ X^\mu_L =& \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\bar{\alpha}^\mu_n e^{-2in(\tau+\sigma)}\,. \end{aligned} \tag{2}$$

This was derived from the gauge fixed action

$$S = \frac{T}{2}\int d^2\sigma (\dot{X}^2 - X'^2)\,. \tag{3}$$

The canonical momentum conjugated to the variable X^μ is given by

$$P^\mu(\sigma,\tau) = \frac{\delta S}{\delta \dot{X}_\mu} = T\dot{X}^\mu\,. \tag{4}$$

The classical Poisson brackets are given by

$$\{P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = 0\,, \quad \{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = 0\,, \quad \{P^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{5}$$

Show that this implies the following Poisson brackets for the modes:

$$\begin{gathered} \{\alpha^\mu_m, \alpha^\nu_n\} = \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\} = -im\eta^{\mu\nu}\delta_{m+n,0} \\ \{\alpha^\mu_m, \bar{\alpha}^\nu_n\} = \{x^\mu, x^\nu\} = \{p^\mu, p^\nu\} = 0, \quad \{x^\mu, p^\nu\} = \eta^{\mu\nu} \\ \{x^\mu, \bar{\alpha}^\nu_n\} = \{p^\mu, \bar{\alpha}^\nu_n\} = \{x^\mu, \alpha^\nu_n\} = \{p^\mu, \alpha^\nu_n\} = 0 \end{gathered} \tag{6}$$

Hint: The Fourier expansion of the Delta distribution on the interval $[0,\pi]$ *reads*

$$\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{2in(\sigma-\sigma')}\,. \tag{7}$$

[1] For simiplicity, we consider here for $\sigma \in [0,\ell], \ell = \pi$.

1.2 Angular momentum & Poincaré algebra (6 Points)

Recall from **Exercise sheet 2**, you derived Angular momentum as the conserved current due to Lorentz transformations. In the conformal gauge it is given by

$$J^{\mu\nu}(\sigma,\tau) = X^\mu P^\nu - X^\nu P^\mu \,. \tag{8}$$

Show that the total Angular momentum can be expressed in terms of the oscillator modes as follows

$$\mathcal{J}^{\mu\nu} = \int_0^\pi d\sigma J^{\mu\nu}(\sigma,\tau) = l^{\mu\nu} + E^{\mu\nu} + \bar{E}^{\mu\nu} \tag{9}$$

where

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \,, \tag{10}$$

$$E^{\mu\nu} = -i \sum_{n\in\mathbb{Z}_{>0}} \frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n)\,, \tag{11}$$

and a similar expression for $\bar{E}^{\mu\nu}$, where we just replace $\alpha^\mu_n \mapsto \bar{\alpha}^\mu_n$. Moreover, from the Poisson brackets in (5) verify that $\mathcal{P}^\mu = \int_0^\pi d\sigma P(\tau,\sigma)$ and $\mathcal{J}^{\mu\nu}$ generate the Poincaré algebra

$$\{\mathcal{P}^\mu, \mathcal{P}^\nu\} = 0\,, \tag{12}$$

$$\{\mathcal{P}^\mu, \mathcal{J}^{\rho\sigma}\} = \eta^{\mu\sigma}\mathcal{P}^\rho - \eta^{\mu\rho}\mathcal{P}^\sigma\,, \tag{13}$$

$$\{\mathcal{J}^{\mu\mu}, \mathcal{J}^{\rho\sigma}\} = \eta^{\mu\rho}\mathcal{J}^{\nu\rho} + \eta^{\nu\rho}\mathcal{J}^{\mu\rho} - \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\sigma}\mathcal{J}^{\nu\rho}\,. \tag{14}$$

2 A lightspeed rotating spaghetti stick (5 Points)

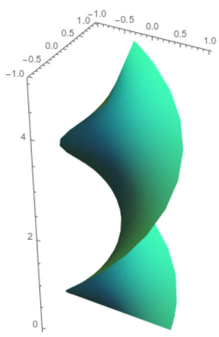

Figure 1: Worldsheet of an open string configuration given in (15). The vertical axis corresponds to the X^0 coordinate. which is orthogonal to the (X^1, X^2) plane.

Here we consider an open string rotating at a constant angular velocity in the (X^1, X^2) plane. Such a configuration is given by

$$X^0 = A\tau, \quad X^1 = A\cos\tau\cos\sigma, \quad X^2 = A\sin\tau\cos\sigma, \quad X^i = 0, \text{ for } i = 3, \ldots D\,. \tag{15}$$

1. Verify that (15) is indeed an open string configuration solution, i.e. that it satisfies the equations of motion.

2. Are the endpoints of the string satisgying Neumann or Dirichlet boundary conditions? Moreover, obtain the speed of the string at its endpoints.
3. Compute the total energy $M \equiv \mathcal{P}^0$ of this configuration.
4. Compute the angular momentum $\mathcal{J} = |\mathcal{J}^{12}|$.
5. Compute[2] $\alpha' \equiv \frac{\mathcal{J}}{M^2}$.

3 The Virasoro algebra (10 Points)

Recall that the energy momentum tensor

$$T_{\mu\nu} = -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h_{\mu\nu}}\,, \tag{16}$$

vanishes in two dimensions. Show that its components in light-cone coordinates

$$\sigma^{\pm} = \tau \pm \sigma\,, \tag{17}$$

are given by

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu\,, \quad T_{--} = \partial_- X^\mu \partial_- X_\mu\,, \tag{18}$$

whereas the other components vanish automatically, i.e.

$$T_{+-} = T_{-+} = 0\,. \tag{19}$$

Show that the Fourier expansions read

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im(\tau-\sigma)}\,, \quad T_{++} = 2l_s^2 \sum_{m=-\infty}^{\infty} \tilde{L}_m e^{-2im(\tau+\sigma)}\,, \tag{20}$$

where the coefficients are given by

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n\,, \quad \tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} \bar{\alpha}_{m-n}\cdot\bar{\alpha}_n\,. \tag{21}$$

Show that the Poisson bracket of two modes is given by

$$\{L_n, L_m\} = i(n-m)L_{n+m}\,. \tag{22}$$

The Virasoro algebra appears also from another - but no unrelated - point of view. The choice of world-sheet metric

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{23}$$

does not fix the diffeomorphism symmetry completely but allows for a redefinition of the light-cone coordinates

$$\sigma^+ \to \sigma^{+'}(\sigma^+)\,, \quad \sigma^- \to \sigma^{-'}(\sigma^-)\,. \tag{24}$$

[2]The parameter α' is the slope of the so-called 'Regge trajectories': the straight line plots of $\mathcal{J}$ vsM^2 seen in nuclear physics in the late 1960s.

1. Show that the action is invariant[3] under

$$\delta X^\mu = a_n e^{2in\sigma^-} \partial_- X^\mu \,. \tag{25}$$

2. Show that the corresponding current is given by

$$j = T \partial_- X^\mu \partial_- X_\mu e^{2in\sigma^-} \,. \tag{26}$$

3. Show that the corresponding charge

$$Q_n = \int d\sigma j^0 \,, \tag{27}$$

is given by L_n.

[3]For simplicity, we just consider a part of the symmetry .