Physikalisches Institut
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Theoretische Physik

Exercise Sheet 4
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WS 18/19

Exercises in Superstring Theory

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`http://www.th.physik.uni-bonn.de/klemm/strings1819/`

Announcement: Due to the holiday *'Allerheiligen'*, the **Group 1** tutorial session will be held on **Wednseday 15:00 ↔ 17:00 in the Besprechungsraum (BCTP)** during this week.

1 A first glimpse at the quantization of the bosonic string (5 Points)

1.1 New coordinates

We consider again the following range of worldsheet coordinates

$$\tau \in [-\infty, \infty]\,, \quad \sigma \in [0, \pi]\,. \tag{1}$$

First we do a Wick rotation, sending

$$\tau \to -i\tau\,. \tag{2}$$

By this the metric on the worldsheet becomes Euclidean. Show that by introducing

$$\zeta = 2(\tau - i\sigma)\,, \quad \bar{\zeta} = 2(\tau + i\sigma)\,, \quad z = e^{\zeta}\,, \quad \bar{z} = e^{\bar{\zeta}}\,, \tag{3}$$

the expansion of the X worldsheet field reads (we just consider one component and therefore omit the spacetime indices)

$$X(z, \bar{z}) = x - i\frac{l_s^2}{4} p \log(|z|^2) + i\frac{l_s}{2} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n z^{-n} + \bar{\alpha}_n \bar{z}^{-n}\right)\,. \tag{4}$$

1.2 Normal ordering ambiguities

We canonically quantize the string by mapping the Fourier modes to operators

$$\alpha_n \mapsto \hat{\alpha}_n\,, \quad \bar{\alpha}_n \mapsto \hat{\bar{\alpha}}_n\,, \tag{5}$$

such that the Poisson bracket is mapped to the commutator

$$\{\cdot, \cdot\} \mapsto i\,[\cdot, \cdot]\,. \tag{6}$$

Therefore one has to deal with normal ordering ambiguities. The normal ordering is defined by[1]

$$\begin{aligned} :xp: &= :px: = xp\,, \quad :\alpha_m \alpha_{-n}: = :\alpha_{-n}\alpha_m: = \alpha_{-n}\alpha_m\,, \\ :\bar{\alpha}_m \bar{\alpha}_{-n}: &= :\bar{\alpha}_{-n}\bar{\alpha}_m: = \bar{\alpha}_{-n}\bar{\alpha}_m\,, \quad m, n \in \mathbb{Z}_{>0}\,. \end{aligned} \tag{7}$$

Show that

$$X(z, \bar{z})X(w, \bar{w}) = :X(z, \bar{z})X(w, \bar{w}): - \frac{l_s^2}{4} \log|z - w|^2\,. \tag{8}$$

[1] In the following we omit the hat for the operators.

2 The Quantum Virasoro Algebra (10 Points)

To fully promote the Virasoro generators to Quantum operators, we need to take into account the normal ordering $: \cdots :$ as introduced in (7)

$$\textbf{Classical: } L_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\alpha_n \mapsto \textbf{Quantum: } \hat{L}_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\hat{\alpha}_{m-n}\hat{\alpha}_n: \,. \tag{9}$$

From now on we drop the hats. Recall that we also follow the replacement (6) and the Fourier modes follow now the following commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}\,, \quad [\alpha^\mu_m, \alpha^\nu_n] = [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\,, \quad [\alpha^\mu_m, \bar{\alpha}^\nu_n] = 0\,. \tag{10}$$

1. Verify that $[L_m, L_n] = (m-n)L_{m+n}$ for $m+n \neq 0$.
2. Show that $[L_m, L_n] - (m-n)L_{m+n} \neq 0$ for $m+n = 0$.

From this result, we note that normal ordering effects appear when $m+n=0$. This modification in the Quantum Virasoro Algebra can be understood as a *central extension* of the classical Virasoro algebra. A central extension $\hat{\mathfrak{g}} \simeq \mathfrak{g} \oplus \mathrm{c}\mathbb{C}$ of a Lie algebra $\mathfrak{g}$ by c satisfies

- $[X,Y]_{\hat{\mathfrak{g}}} = [X,Y]_{\mathfrak{g}} + \mathrm{c}P(X,Y)\,, \quad X, Y \in \mathfrak{g}\,,$
- $[X, \mathrm{c}]_{\hat{\mathfrak{g}}} = 0\,,$
- $[\mathrm{c}, \mathrm{c}]_{\hat{\mathfrak{g}}} = 0\,,$

i.e. c belongs to the center of $\hat{\mathfrak{g}}$. Here $P : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is bilinear and antisymmetric. Recall $[L_m, L_n]$ for $m+n=0$, any ambiguity here arising due to normal ordering would only involve a complex number, therefore we are guaranteed to have

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n}\,. \tag{11}$$

In other words the central extension term is $\mathrm{c}P(L_m, L_n) = A(m)\delta_{m+n}$.

3. Show that
$$A(m) = \frac{1}{12}D(m^3 - m)\,. \tag{12}$$
Hint: First obtain that $A(-m) = -A(m)$. Then by using the Jacobi-identity derive $A(m) = c_1 m + c_3 m^3$. Determine c_1 and c_3 by evaluating $[L_2, L_{-2}]$ on the vacuum state.

3 Lorentz symmetry of the quantum string (10 Points)

On the first sheet you found the currents $J^{\mu\nu}$ associated to Lorentz symmetry. The conserved charge

$$\mathcal{J}^{\mu\nu} = \int d\sigma J^{\mu\nu}\,, \tag{13}$$

becomes a (normal-ordered) operator in the quantum theory. Show that

$$[L_m, \mathcal{J}^{\mu\nu}] = 0\,. \tag{14}$$

What does this imply for the spectrum concerning representations of the Lorentz group?