Physikalisches Institut
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Exercise Sheet 5
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Exercises in Superstring Theory

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`http://www.th.physik.uni-bonn.de/klemm/strings1819/`

1 Physical states and spurious states (10 Points)

We start by recalling some definitions. [1] A state $|\eta\rangle$ satisfies the mass-shell condition, if

$$(L_0 - a)|\eta\rangle = 0\,. \tag{1}$$

Here L_0 is a normal-ordered operator and a is a constant that takes into account the freedom in substracting an arbitrary constant in the normal ordering process and needs to be fixed later. A state $|\phi\rangle$ is called physical if it satisfies the mass-shell condition and furthermore obeys

$$L_m|\phi\rangle = 0\,, \quad \forall m > 0\,. \tag{2}$$

A state $|\psi\rangle$ is called spurious, if it is orthogonal to all physical states. Note that by definition a state that is spurious *and* physical has zero norm. Those states are called null states. Next we consider the first excited state in open string theory which is given by

$$|\zeta, k\rangle = \zeta \cdot \alpha_{-1}|0, k\rangle\,. \tag{3}$$

Here $|0, k\rangle$ is the vacuum, i.e. an unexcited string having momentum k^μ and ζ^μ denotes a polarization vector. The mass of such a state is given by

$$M^2 = \frac{1}{l_s^2}(N - a)\,, \quad N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n\,, \tag{4}$$

where N is called the number operator.

1. Show that

$$N\,\alpha_{-n_1} \cdot \ldots \cdot \alpha_{-n_k}|0, k\rangle = \left(\sum_{i=1}^{k} n_i\right) \alpha_{-n_1} \cdot \ldots \cdot \alpha_{-n_k}|0, k\rangle\,, \tag{5}$$

where $n_i > 0$.

2. Show that the state

$$|\psi\rangle = L_{-1}|0, k\rangle \tag{6}$$

is spurious.

[1] The same approach can be applied to electrodynamics and is known as Gupta-Bleuler quantization.

3. Consider the cases $a < 1$, $a = 1$ and $a > 1$ and comment on
   - the mass of $|\zeta, k\rangle$
   - if $|\psi\rangle$ is physical.

   Which is the right value for a in order to have a massless vector in the spectrum?

The null states decouple from physical processes as they are orthogonal to all physical states. Therefore one may identify

$$|\phi\rangle \sim |\phi\rangle + |\eta\rangle\,, \quad \forall\,|\eta\rangle\,\text{null}\,. \tag{7}$$

4. Show that the condition

$$L_n|\zeta, k\rangle = 0\,, \quad n > 0\,, \tag{8}$$

   is satisfied automatically for $n > 1$ and for $n = 1$ leads to

$$k^\mu \zeta_\mu |0, k\rangle = 0\,. \tag{9}$$

5. What is the condition

$$k^\mu \zeta_\mu = 0\,, \tag{10}$$

   in position space?

6. Show for the null state

$$L_{-1}|0, k\rangle = \sqrt{2\alpha'} k_\mu \alpha_{-1}^\mu |0, k\rangle\,. \tag{11}$$

7. What is therefore the interpretation of the identification (7) in position space?

2 Determination of a and D (5 Points)

In the previous exercise we have seen the physical relevance of null states. An important question is therefore under which conditions the theory contains "a lot" of these states. We consider states of the form

$$|\psi\rangle = L_{-1}|\chi_1\rangle\,, \quad (L_0 - a + 1)|\chi_1\rangle = 0\,, \quad L_m|\chi_1\rangle = 0\,, \quad m > 0\,. \tag{12}$$

1. Show that the physical state condition implies in the case $m = 1$

$$L_1|\psi\rangle = 2(a - 1)|\chi_1\rangle = 0\,. \tag{13}$$

   Hint: Use the virasoro algebra.

Next we consider states of the form

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}^2)|\bar{\chi}\rangle\,, \quad (L_0 + 1)|\bar{\chi}\rangle = L_m|\bar{\chi}\rangle = 0\,, \quad m > 0\,. \tag{14}$$

2. Show that

$$L_1|\psi\rangle = 0 \tag{15}$$

   leads to $\gamma = \frac{3}{2}$.

3. Show that

$$L_2|\psi\rangle = \left(-13 + \frac{D}{2}\right)|\bar{\chi}\rangle\,, \tag{16}$$

   and conclude that $D = 26$.

3 A different way to fix D (5 Points)

Another way to fix the number of space-time dimensions D, is given by evaluating the normal ordering constant a of L_0 by applying the so-called ζ-function regularization. The ζ-function can be expanded in a region of the complex plane as

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}\,, \quad \mathrm{Re}\, s > 1\,. \tag{17}$$

It can be analytically continued to a meromorphic function on the whole complex plane. In particular it obeys

$$\zeta(-1) = -\frac{1}{12}\,. \tag{18}$$

1. Show that a is formally given as

$$a = -\frac{1}{2}\sum_{m=1}^{\infty}\left[a_m^i, a_{-m}^j\right]\delta_{ij}\,. \tag{19}$$

2. Show that this fixes D by treating the infinite sum as a value of the ζ-function. Hint: So far we have been using the light-cone gauge quantization. Therefore there are just $D-2$ oscillation modes.

4 Spectrum of the quantized bosonic string (10 Points)

In the previous exercises you determined the normal ordering constant $a = 1$ and the spacetime dimension $D = 26$ for critical bosonic string theory. In the following, consider light-cone gauge quantization of bosonic string theory.

Recall that $(L_0 - a)\,|\phi\rangle = 0$ for a physical state $|\phi\rangle$ in open bosonic string theory (with NN boundary conditions) corresponds to the mass-shell condition

$$\alpha' M^2 = -p_\mu p^\mu = 2p^+p^- - \sum_{i=1}^{24} p_i^2 = N - 1\,. \tag{20}$$

For a physical state $|\phi\rangle$ in closed bosonic string theory, we have $(L_0 - a)\,|\phi\rangle = (\bar{L}_0 - a)\,|\phi\rangle = 0$, which corresponds to the mass-shell condition

$$\alpha' M^2 = -p_\mu p^\mu = 2p^+p^- - \sum_{i=1}^{24} p_i^2 = 4(N-1) = 4(\bar{N}-1)\,. \tag{21}$$

The number operators are given by

$$N = \sum_{i=1}^{24}\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i\,, \qquad \bar{N} = \sum_{i=1}^{24}\sum_{n=1}^{\infty} \bar{\alpha}_{-n}^i \bar{\alpha}_n^i\,. \tag{22}$$

1. Find the states for the first three levels (including the ground state) in the spectrum of open bosonic string and count the number of states corresponding to their $SO(24)$ representations for each level.

Hint: Remember the useful Young tableau from previous courses to count the number of states ($\equiv$ dimension of the representation).

Recall that massless states are classified by representations of $SO(24)$, whereas massive states are classified by representations of $SO(25)$. These are the *little groups* that appear in bosonic string theory.

2. Calculate the masses for each of the states of item 1. and see into which little group their representations should fall. Count the number of states corresponding to their little group representations for each level and confirm that they match with the number of states you found in item 1.[2].

3. What does $(L_0 - a)\,|\phi\rangle = (\bar{L}_0 - a)\,|\phi\rangle = 0$ imply for N and $\bar{N}$?

4. Find the states for the first two levels (including the ground state) in the spectrum of closed bosonic string (use the *level-matching condition* from item 3., which relates left to right moving modes) and count the number of states corresponding to their $SO(24)$ representations for each level.

5. Calculate the masses for each of the states of item 4. and see into which little group their representations should fall. Count the number of states corresponding to their little group representations for each level and confirm that they match with the number of states you found in item 4. For the second state ($\equiv$ the first excited state), separate it in: symmetric and traceless; antisymmetric; and trace parts; and comment on the physical interpretation of each part.

[2] Of course, if the little group is $SO(24)$, the number of states match trivially.