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Exercises in Superstring Theory

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1 The conformal group in d dimensions

Consider the Euclidean d-dimensional spacetime $\mathbb{R}^{d,0}$ with metric $\eta_{\mu\nu} = \mathrm{diag}(1, ..., 1)$, $\mu,\nu = 1, ..., d$. By definition, a conformal transformation of coordinates leaves the metric tensor invariant up to a scale, i.e.,

$$\eta'_{\mu\nu}(x'^{\mu}) = \eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x^{\mu}) \eta_{\mu\nu}(x^{\mu}) \ . \tag{1.1}$$

It preserves angles between any two arbitrary vectors on spacetime.

In this exercise, you will familiarize yourself with the conformal group in d dimensions and its algebra, noting that it contains the Poincaré group as a subgroup (when $\Lambda(x^{\mu}) = 1$).

1. Show that the consequence of requiring that an infinitesimal coordinate transformation

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x^{\mu}), \ \epsilon(x^{\mu}) \ll 1, \tag{1.2}$$

is conformal (i.e., that it satisfies (1.1)) leads to

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = f \eta_{\mu\nu} \ , \tag{1.3}$$

where

$$f = \frac{2}{d}(\partial \cdot \epsilon), \qquad \partial \cdot \epsilon = \partial_{\mu}\epsilon^{\mu} \ . \tag{1.4}$$

2. Take the partial derivative ∂_{ρ} of (1.3), permute the indices of this resulting equation to find two similar equations. Now take a convenient linear combination of these three equations to find

$$2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \eta_{\nu\rho}\partial_{\mu} f + \eta_{\rho\mu}\partial_{\nu} f - \eta_{\mu\nu}\partial_{\rho} f \ . \tag{1.5}$$

3. Contract (1.5) with $\eta^{\mu\nu}$ and take ∂_{ν} of the resulting expression. Moreover, take ∂^2 of (1.3). Combine these results to get

$$(2 - d)\partial_{\mu}\partial_{\nu} f = \eta_{\mu\nu}\partial^2 f \ . \tag{1.6}$$

Contracting (1.6) further with $\eta^{\mu\nu}$ leads to

$$(d - 1)\partial^2 f = 0 \ . \tag{1.7}$$

From (1.7), one clearly sees that, for $d = 1$, there is no constraint on the function f. This means that any transformation in one dimension is conformal[1]. The 2-dimensional case will be studied in the next exercise. Let us now focus on the case $d \geq 3$.

[1] In fact the notion of angle does not even exist in one dimension.

4. Equations (1.6) and (1.7) imply that $\partial_\mu \partial_\nu f = 0$, i.e., f is a linear function in the coordinates x^μ, $f(x^\mu) = A + B_\mu x^\mu$. Explain why this condition on f implies that ϵ_μ can be written as

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \ , \quad c_{\mu\nu\rho} = c_{\mu\rho\nu} \ , \tag{1.8}$$

where conditions on the coefficients a_μ, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ will be determined below.

Since (1.3) – (1.5) hold for all x^μ, we can treat each power of x^μ in (1.8) separately.

5. Show that:

(i) there are no constraints on the constant term a_μ;

(ii) substitution of the linear term of (1.8) in (1.3) implies

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\rho{}_\rho \eta_{\mu\nu} \ ; \tag{1.9}$$

(iii) substitution of the quadratic term of (1.8) in (1.5) implies

$$c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu \ , \qquad b_\mu := \frac{1}{d} c^\sigma{}_{\sigma\mu} \ . \tag{1.10}$$

The term a_μ gives rise to an infinitesimal translation.
Moreover, (1.9) implies that b_μ can be separated into the sum of an antisymmetric part and a pure trace part as follows

$$b_{\mu\nu} = m_{\mu\nu} + \alpha \eta_{\mu\nu} \ , \qquad m_{\mu\nu} = -m_{\nu\mu} \ . \tag{1.11}$$

The antisymmetric part gives rise to infinitesimal rotations whereas the pure trace part gives rise to an infinitesimal scale transformation.
The infinitesimal transformation associated to $c_{\mu\nu\rho}$ is given by

$$x'^\mu = x^\mu + 2(b \cdot x) x^\mu - b^\mu x^2 \tag{1.12}$$

and it receives the name of *special conformal transformation* (SCT).
To each infinitesimal transformation, one gets a finite one, from which the generators of the conformal group can be identified.

The table below summarizes the finite conformal transformations together with the corresponding generators of the conformal group (translations and rotations form the usual Poincaré group).

Transformations		**Generators**
Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Rotation	$x'^\mu = M^\mu_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
Dilation	$x'^\mu = \alpha x^\mu$	$D = -i x^\mu \partial_\mu$
SCT	$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$	$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

The generators of the conformal group obey the conformal algebra given below

$$\begin{aligned}
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\
&[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\
&[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
&[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})
\end{aligned} \tag{1.13}$$

6. Check the first four relations of (1.13). Use $[x_\mu, P_\nu] = i\eta_{\mu\nu}$.

In order to put the conformal algebra above into a simpler form, we define the following generators

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} , \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) , \\ J_{-1,0} &= D , \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) . \end{aligned} \tag{1.14}$$

7. Show that the generators above satisfy the algebra of $SO(d+1,1)$, i.e.,

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) , \tag{1.15}$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, ..., 1)$.

In the similar case of Minkowski spacetime $\mathbb{R}^{d-1,1}$, where $\eta_{ab} = \text{diag}(-1, -1, 1, ..., 1)$, the commutation relations satisfy the algebra of $SO(d, 2)$.

2 Two-point & three-point correlation functions

A field $\phi(z, \bar{z})$ is called a *primary field* of conformal dimension[2] $(h, \bar{h})$ if, under conformal transformations, $z \to z' = f(z)$, $\bar{z} \to \bar{z}' = f(\bar{z})$, it transforms as

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) . \tag{2.1}$$

If (2.1) holds only for global conformal transformations, $z', \bar{z}' \in SL(2, \mathbb{C})/\mathbb{Z}_2$, then ϕ is called a *quasi-primary field* [3]. Note that a primary field is always quasi-primary, but the reverse is not true. A field which is not primary or quasi-primary is called *secondary.*

Restricting only to the holomorphic part (also called chiral), a primary field transforms, under infinitesimal conformal transformations $z \to z + \epsilon(z)$, $\epsilon(z) \ll 1$, as

$$\phi(z) \to \phi'(z) = \phi(z) + \delta_\epsilon \phi(z) \qquad \text{with} \qquad \delta_\epsilon \phi(z) = -[\epsilon(z)\partial_z + h\partial_z \epsilon(z)]\phi(z) . \tag{2.2}$$

Consider the two-point correlation function of two quasi-primary fields $\phi_i = \phi_i(z_i)$ with conformal dimension h_i, $i = 1, 2$, in 2d CFT: $\quad G(z_1, z_2) = \langle \phi_1(z_1)\phi_2(z_2) \rangle$.

1. Show that conformal invariance of $G(z_1, z_2)$ implies

$$[\epsilon(z_1)\partial_{z_1} + h_1\partial_{z_1}\epsilon(z_1) + \epsilon(z_2)\partial_{z_2} + h_2\partial_{z_2}\epsilon(z_2)]G(z_1, z_2) = 0 . \tag{2.3}$$

Hint: Recall $\delta_\epsilon \phi(z)$ from (2.2) and see what happens for $\delta_{\epsilon(z_1),\epsilon(z_2)}$ on $G(z_1, z_2)$.

For infinitesimal global conformal transformations one can show that $\epsilon(z) = \alpha + \beta z + \gamma z^2$ at first order in $\epsilon(z)$, where α, β and γ are constant infinitesimal parameters.

2. Use $\epsilon(z_i) = \alpha$ in (2.3) to show that $G(z_1, z_2)$ depends on $x = z_1 - z_2$ only.
3. Use $\epsilon(z_i) = \beta z_i$ in (2.3) to show that $G(z_1, z_2)$ is written as

$$G(z_1, z_2) = \frac{C_{12}}{(z_1 - z_2)^{h_1+h_2}} . \tag{2.4}$$

where C_{12} is a constant.

[2] Also called weight or scaling dimension.
[3] Later in the course, you will see that the energy-momentum tensor is an example of a quasi-primary field.

4. Use $\epsilon(z_i) = \gamma z_i^2$ in (2.3) to show that $G(z_1, z_2)$ vanishes unless $h_1 = h_2$.

In other words, two-point functions are given by

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \begin{cases} \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}} & h_1 = h_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

In the same way, one shows that three-point functions are given by

$$\langle \phi_1(z_1)\phi_2(z_2)\phi(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1}} \ , \tag{2.6}$$

where C_{123} is a constant and $z_{ij} = z_i - z_j$, $i = 1, 2, 3$.